SOLITON DYNAMICS FOR THE NONLINEAR SCHRÖDINGER EQUATION WITH MAGNETIC FIELD

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ABSTRACT. The semiclassical regime of a nonlinear focusing Schrödinger equation in presence of non-constant electric and magnetic potentials V, A is studied by taking as initial datum the ground state solution of an associated autonomous stationary equation. The concentration curve of the solutions is a parameterization of the solutions of the second order ordinary equation $\ddot{x} = -\nabla V(x) - \dot{x} \times B(x)$, where $B = \nabla \times A$ is the magnetic field of a given magnetic potential A.

CONTENTS

1. INTRODUCTION

The aim of this paper is the study of the asymptotic behaviour of the solutions of the semilinear Schrödinger equation with an external magnetic potential A,

$$(P) \qquad \begin{cases} \mathrm{i}\varepsilon\partial_t\phi_\varepsilon = \frac{1}{2}\big(\frac{\varepsilon}{\mathrm{i}}\nabla - A(x)\big)^2\phi_\varepsilon + V(x)\phi_\varepsilon - |\phi_\varepsilon|^{2p}\phi_\varepsilon, & x \in \mathbb{R}^N,\ t > 0, \\ \phi_\varepsilon(x,0) = \phi_0(x), & x \in \mathbb{R}^N, \end{cases}$$

in the semiclassical regime of ε going to zero, by choosing a suitable class of initial data ϕ_0 which is related to the (unique) ground state solution r of an associated elliptic problem.

2000 *Mathematics Subject Classification.* 83C50, 81Q05, 35Q40, 35Q51, 35Q55, 37K40, 37K45.

Key words and phrases. Nonlinear Schrödinger equation, magnetic fields, soliton dynamics, concentration phenomena, semiclassical limit, stability of ground states.

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The author was supported by the 2007 MIUR national research project entitled *"Variational and Topological Methods in the Study of Nonlinear Phenomena"*.

We will show that the evolution $\phi_\varepsilon(t)$ remains close to r, in a suitable sense (and with an *explicit* convergence rate), locally uniformly in time, in the transition from quantum to classical mechanics, namely as ε vanishes. This dynamical behaviour is also known as *soliton dynamics* (for a beautiful survey on *solitons* and their stability, see [54]). Here, i is the imaginary unit, ε is a small positive parameter playing the rôle of Planck's constant, $N \geq 1$, $0 < p < 2/N$ and $V : \mathbb{R}^N \to \mathbb{R}$, $A : \mathbb{R}^N \to \mathbb{R}^N$ are an *electric* and *magnetic* potentials respectively. The magnetic field B is $B = \nabla \times A$ in $\mathbb{R}^3$ and can be thought (and identified) in general dimension as a 2-form $\mathbb{H}^B$ of coefficients $(\partial_i A_j - \partial_j A_i)$. The magnetic Schrödinger operator which appears in problem (P) formally operates as follows

$$\left(\frac{\varepsilon}{\mathrm{i}}\nabla - A(x)\right)^2\phi = -\varepsilon^2\Delta\phi - \frac{2\varepsilon}{\mathrm{i}}A(x)\cdot\nabla\phi + |A(x)|^2\phi - \frac{\varepsilon}{\mathrm{i}}\mathrm{div}_x A(x)\phi, \tag{1.1}$$

and it has been intensively studied in works by J. Avron, I. Herbst and B. Simon around 1978 (see [4, 5, 6, 45, 49] and references therein). If $A = 0$, then equation (P) reduces to

$$\begin{cases} \mathrm{i}\varepsilon\partial_t\phi_\varepsilon = -\frac{\varepsilon^2}{2}\Delta\phi_\varepsilon + V(x)\phi_\varepsilon - |\phi_\varepsilon|^{2p}\phi_\varepsilon, & x \in \mathbb{R}^N,\ t > 0, \\ \phi_\varepsilon(x,0) = \phi_0(x), & x \in \mathbb{R}^N. \end{cases} \tag{1.2}$$

For equation (1.2), rigorous results about the soliton dynamics were obtained in various papers by J.C. Bronski, R.L. Jerrard [9] and S. Keraani [38, 39] via arguments purely based on the use of conservation laws satisfied by the equation and by the associated Hamiltonian system in $\mathbb{R}^N$ built upon the potential V, that is the *Newton law*

$$\ddot{x} = -\nabla V(x), \quad \dot{x}(0) = \xi_0,\ x(0) = x_0. \tag{1.3}$$

For other achievements about the full dynamics of (1.2), see also [30, 31] (in the framework of orbital stability of standing waves) as well as [36, 37] (in the framework of non-integrable perturbation of integrable systems). Similar results were investigated in geometric optics by a different technique (WKB method), namely writing formally the solution as $u_\varepsilon = U_\varepsilon(x,t)e^{\mathrm{i}\theta(x,t)/\varepsilon}$, where $U_\varepsilon = U_0 + \varepsilon U_1 + \varepsilon^2 U_2 \cdots$, where θ and U_j are solutions, respectively, of a Hamilton-Jacobi type equation (known as *eikonal* equation) and of a system of transport equations.

It is very important to stress that, in the particular case of *standing wave solutions* of (1.2), namely special solutions of (1.2) of the form

$$\phi_\varepsilon(x,t) = u_\varepsilon(x)e^{-\frac{\mathrm{i}}{\varepsilon}\theta t}, \quad x \in \mathbb{R}^N,\ t \in \mathbb{R}^+, \quad (\theta \in \mathbb{R}),$$

where $u_\varepsilon : \mathbb{R}^N \to \mathbb{R}$, there is an enormous literature regarding the semiclassical limit for the corresponding elliptic equation

$$-\tfrac{\varepsilon^2}{2}\Delta u_\varepsilon + V(x)u_\varepsilon = |u_\varepsilon|^{2p}u_\varepsilon, \quad x \in \mathbb{R}^N.$$

See the recent book [2] by A. Ambrosetti and A. Malchiodi and the references therein. Moreover, there are various works admitting the presence of a magnetic potential A, and studying the asymptotic profile of the solutions $u_\varepsilon : \mathbb{R}^N \to \mathbb{C}$ to

$$\tfrac{1}{2}\left(\tfrac{\varepsilon}{\mathrm{i}}\nabla - A(x)\right)^2 u_\varepsilon + V(x)u_\varepsilon = |u_\varepsilon|^{2p}u_\varepsilon, \quad x \in \mathbb{R}^N,$$

as ε goes to zero (see e.g. [3, 7, 18, 19, 20, 21, 40, 47] and references therein).

In the special case $A = V = 0$, the orbital stability for problem (1.2) was proved by T. Cazenave and P.L. Lions [16], and by M. Weinstein in [58, 59]. Then, A. Soffer and M. Weinstein proved in [50] the asymptotic stability of nonlinear ground states of (1.2).

See also the following seminal contributions (*in alphabetical order*): W.K. Abou Salem [1], V. Buslaev and G. Perelman [10, 11], V. Buslaev and C. Sulem [12], J. Fröhlich, S. Gustafson, L. Jonsson, I.M. Sigal, T.-P. Tsai and H.-T. Yau [25, 26, 27, 28], S. Gustafson and M.I. Sigal [32], J. Holmer and Zworski [33, 34], A. Soffer and M. Weinstein [51, 52], T.-P. Tsai and H.-T. Yau [55, 56, 57]. See also the references included in these works.

Now, in presence of a magnetic, some natural questions arise: what is the rôle played by the magnetic field B? if B plays a significant rôle, what is the correct Newton equation taking the place of (1.3), which characterizes the concentrating curve and drives the dynamic in the semiclassical limit?

As known, a charged particle moving in a magnetic field B feels a sideways force that is proportional to the strength of B as well as to its velocity. This force, which is always perpendicular to both the velocity of the particle and the magnetic field that created it (a particle moving in the direction of B does not experience a force) is known as the *Lorentz force*. Hence, charged particles move in a circle (or more generally, *helix*) around the field lines of B (*cyclotron motion*). During the motion, B can do no work on a charged particle (cannot speed it up or slow it down) although it changes its direction (See figures 1 and 2).

As a consequence, with the expectation (which arises from the magnetic-free case) that in the semiclassical limit the dynamics is governed by the classical Newtonian law, one is tempted to say that, in presence of an external magnetic field B, the right counterpart of (1.3) is given by the following Newton equation

$$\ddot{x} = -\nabla V(x) - \dot{x} \times B(x), \quad \dot{x}(0) = \xi_0, \ x(0) = x_0, \tag{1.4}$$

agreeing that $\times$ has to be interpreted as a matrix operation ($\mathbb{H}^B \dot{x}$) if we are not in $\mathbb{R}^3$.

Only after full completion of the present paper the author discovered that a first result (mass and momentum asymptotics) in this direction was obtained, independently, with decay assumptions on B , by A. Selvitella in [48], showing that, in fact, the above guess is the correct interpretation, in the transition process from quantum to classical mechanics.

On the other hand, in this paper, we improve the result of [48], proving a stronger result, which is precisely the one predicted by the WKB method. Roughly speaking, under suitable regularity assumptions on V and A, we show that, given the initial position and velocity x_0, ξ_0 in $\mathbb{R}^N$, and taking as initial datum for (P)

$$\phi_0(x) = r\Big(\frac{x - x_0}{\varepsilon}\Big) e^{\frac{\mathrm{i}}{\varepsilon}[A(x_0)\cdot(x-x_0)+x\cdot\xi_0]}, \quad x \in \mathbb{R}^N, \tag{I}$$

where $r \in H^1(\mathbb{R}^N)$ is the unique (up to translation) real ground state solution (bump like) of the associated elliptic problem

$$-\frac{1}{2}\Delta r + r = |r|^{2p} r \qquad \text{in } \mathbb{R}^N, \tag{S}$$

then there exists a family of shift functions $\theta_\varepsilon : \mathbb{R}^+ \to [0, 2\pi)$ such that

$$\phi_\varepsilon(x,t) = r\Big(\frac{x - x(t)}{\varepsilon}\Big) e^{\frac{\mathrm{i}}{\varepsilon}[A(x(t))\cdot(x-x(t))+x\cdot\dot{x}(t)+\theta_\varepsilon(t)]} + \omega_\varepsilon, \quad x \in \mathbb{R}^N, \ t > 0, \tag{1.5}$$

locally uniformly in time, as ε goes to zero, where we have set $\|\omega_\varepsilon\|_{\mathbb{H}_\varepsilon} = \mathcal{O}(\varepsilon)$, and being $\|\phi\|^2_{\mathbb{H}_\varepsilon} = \varepsilon^{2-N}\|\nabla\phi\|^2_{L^2} + \varepsilon^{-N}\|\phi\|^2_{L^2}$. In particular, with respect to [48], the convergence rate is *explicit and of the order* ε and, as a direct consequence, the concentration center in the representation formula (1.5) (expressing the soliton dynamics as guessed by the WKB

method) is *exactly* $x(t)$ (in [48] formula (1.5) is not achievable, being the convergence rate undetermined).

The magnetic potential A contributes to the *phase* of the solution, and $x(t)$ is the *concentration line* (which can be considerably influenced by the presence of B, see the phase portraits in figures 1-2). Initial data (I) should also be thought as corresponding to a *point particle* with position x_0 and velocity ξ_0.

In the case where $\xi_0 = 0$ and x_0 is a *critical point* of the potential V, as equation (1.4) admits the trivial solution $x(t) = x_0$ and $\xi(t) = 0$ for all $t \in \mathbb{R}^+$, formula (1.5) reduces to

$$\phi_\varepsilon(x,t) = r\Big(\frac{x-x_0}{\varepsilon}\Big)e^{\frac{\mathrm{i}}{\varepsilon}[A(x_0)\cdot(x-x_0)+\theta_\varepsilon(t)]} + \omega_\varepsilon, \quad x \in \mathbb{R}^N,\ t > 0,$$

locally uniformly in time, as ε goes to zero (see Remark 2.5). In turn, the concentration of ϕ_ε is *static* and takes place around the critical points of V, instead occurring along a smooth curve in $\mathbb{R}^N$. This is consistent with the literature for the standing wave solutions mentioned above.

Acknowledgements.

The author wish to thank Proff. I.M. Sigal, E.H. Lieb, T.-P. Tsai and S. Keraani for pointing out useful information and bibliographic references related to the soliton dynamic of NLS equations. The author also thanks Proff. M. Degiovanni and D. Fortunato for feedbacks on a preliminary version of the paper. The author was supported by the 2007 MIUR national research project entitled: *"Variational and Topological Methods in the Study of Nonlinear Phenomena"*.

Organization of the paper.

In Section 2, we introduce the functional framework, the tools and the ingredients needed to write the statement of the main result of the paper, Theorem 2.4. In Section 3, we collect various preparatory results concerning the characterization of the energy levels of the problem, in the semiclassical regime. In Section 4, we state the main approximation estimates for the solutions. In Section 5, we get two integral identities for the evolution of the mass and momentum densities. In Section 6, we obtain the approximation results for the mass and momentum densities. In Section 7, we obtain an error estimate. In turn, we conclude the proof of the main result of the paper, Theorem 2.4. Finally, In Section 8, we summarize the results obtained.

Main notations.

(1) The imaginary unit is denoted by i.
(2) The complex conjugate of any number $z \in \mathbb{C}$ is denoted by $\bar{z}$.
(3) The real part of a number $z \in \mathbb{C}$ is denoted by $\Re z$.
(4) The imaginary part of a number $z \in \mathbb{C}$ is denoted by $\Im z$.
(5) For all $z, w \in \mathbb{C}$ it holds $\Re(\bar{z}w) = \Re(z\bar{w})$.
(6) For all $z, w \in \mathbb{C}$ it holds $\Im(\bar{z}w) = -\Im(z\bar{w})$.
(7) The symbol $\mathbb{R}^+$ means the positive real line $[0, \infty)$.

(8) The ordinary inner product between two vectors $a, b \in \mathbb{R}^N$ is denoted by $a \cdot b$.
(9) The standard L^2 norm of a function u is denoted by $\|u\|_{L^2}$.
(10) The standard L^∞ norm of a function u is denoted by $\|u\|_{L^\infty}$.
(11) The symbols ∂_t and ∂_j mean $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial x_j}$ respectively.
(12) The symbol Δ means $\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_N^2}$.
(13) The symbol $C^m(\mathbb{R}^N)$, for $m \in \mathbb{N}$, denotes the space of functions with continuous derivatives up to the order m. Sometimes $C^m(\mathbb{R}^N)$ is endowed with the norm

$$\|\phi\|_{C^m} = \sum_{|\alpha| \le m} \|D^\alpha \phi\|_{L^\infty} < \infty.$$

(14) The symbol $\int f$ stands for the integral of f over $\mathbb{R}^N$ with the Lebesgue measure.
(15) The symbol C^{2*} denotes the dual space of C^2. The norm of a ν in C^{2*} is

$$\|\nu\|_{C^{2*}} = \sup\Big\{\Big| \int \phi(x)\nu dx\Big| : \phi \in C^2(\mathbb{R}^N),\ \|\phi\|_{C^2} \le 1\Big\}.$$

Clearly, C^{2*} contains the space of bounded Radon measures.
(16) C denotes a generic positive constant, which may vary inside a chain of inequalities.
(17) The first and second ordinary derivatives of $t \mapsto x(t)$ are denoted by $\dot{x}$ and $\ddot{x}$.
(18) We use the Landau symbols. In particular $\mathcal{O}(\varepsilon)$ is a generic function such that the $\limsup$ of $\varepsilon^{-1}\mathcal{O}(\varepsilon)$ is finite, as ε goes to zero.

2. Statement of the main result

2.1. **Functional setup and tools.** It is quite natural to consider operator (1.1) on the Hilbert space $H_{A,\varepsilon}$ defined by the closure of $C_c^\infty(\mathbb{R}^N;\mathbb{C})$ under the scalar product

$$(u, v)_{H_{A,\varepsilon}} = \Re \int (D^\varepsilon u \cdot \overline{D^\varepsilon v} + V(x)u\overline{v})dx,$$

where $D^\varepsilon u = (D_1^\varepsilon u, \dots, D_N^\varepsilon u)$ and $D_j^\varepsilon = \mathrm{i}^{-1}\varepsilon\partial_j - A_j(x)$, with induced norm

$$\|u\|_{H_{A,\varepsilon}}^2 = \int \Big|\frac{\varepsilon}{\mathrm{i}}\nabla u - A(x)u\Big|^2 dx + \int V(x)|u|^2 dx < \infty.$$

The dual space of $H_{A,\varepsilon}$ is denoted by $H'_{A,\varepsilon}$, while the space $H^2_{A,\varepsilon}$ is the set of u such that

$$\|u\|_{H^2_{A,\varepsilon}}^2 = \|u\|_{L^2}^2 + \|\big(\frac{\varepsilon}{\mathrm{i}}\nabla - A(x)\big)^2 u\|_{L^2}^2 < \infty.$$

Moreover, to problem (P) it can be naturally associated the functional $E : H_{A,\varepsilon} \to \mathbb{R}$ (see also formula (2.4))

$$E(u) = \frac{1}{2}\int \Big|\frac{\varepsilon}{\mathrm{i}}\nabla u - A(x)u\Big|^2 dx + \int V(x)|u|^2 dx - \frac{1}{p+1}\int |u|^{2p+2}dx.$$

Finally, we consider the functional $\mathcal{E} : H^1(\mathbb{R}^N;\mathbb{C}) \to \mathbb{R}$ associated with (S)

$$\mathcal{E}(u) = \frac{1}{2}\int |\nabla u|^2 dx - \frac{1}{p+1}\int |u|^{2p+2}dx.$$

It is a standard fact that the solution r of (S) is the unique (up to translation) solution of the following minimization problem

$$\mathcal{E}(r) = \min\{\mathcal{E}(u) : u \in H^1(\mathbb{R}^N),\ \|u\|_{L^2} = \|r\|_{L^2}\}. \tag{2.1}$$

We also set

$$m := \|r\|^2_{L^2}. \tag{2.2}$$

Also, r is radially symmetric and decreasing, belongs to $C^2(\mathbb{R}^N) \cap H^2(\mathbb{R}^N)$, and it decays exponentially together with its derivatives up to the order two, that is

$$|D^\alpha r(x)| \le Ce^{-\sigma|x|}, \quad x \in \mathbb{R}^N, \tag{2.3}$$

for some $\sigma, C > 0$ and all $0 \le |\alpha| \le 2$ (see e.g. [8]).

2.2. **Well-posedness and conservation laws.** We recall that in [15, Section 9.1] (see also [23]), in the particular case $N = 3$ and when the external magnetic field $B = (b_1, b_2, b_3)$ is constant (thus A is linear with respect to x), the (global) well-posedness of problem (P) in the (natural) energy space $H_{A,\varepsilon}$ as well as the $H^2_{A,\varepsilon}$-regularity of the flux for $H^2_{A,\varepsilon}$-initial data was investigated (see Proposition 2.2 below) by T. Cazenave, M. Esteban and P.L. Lions. Furthermore, in general dimension N and for a general (smooth) vector potential A, the (local) well-posedness in the energy space $H_{A,\varepsilon}$ has been recently studied in [42] by L. Michel. We also wish to cite earlier papers by Y. Nakamura and A. Shimomura [43], Y. Nakamura [44] as well as the important paper by K. Yajima [60]. In particular, in [43], if B has decay assumptions at infinity, the problem is locally solved in the weighted space $\Sigma(2) \subset H^2(\mathbb{R}^N;\mathbb{C})$ of functions f in $L^2(\mathbb{R}^N;\mathbb{C})$ such that $\|x^\alpha D^\beta f\|_{L^2} < \infty$ for all α, β with $|\alpha|, |\beta| \ge 0$ and $0 \le |\alpha + \beta| \le 2$ (notice that, via the decay (2.3), the initial datum ϕ_0 in (I) belongs to the space $\Sigma(2)$). Finally, see also [15, Theorems 4.6.5 and 5.5.1] and an abstract result, Lemma A.1, in the Appendix of [17], by T. Cazenave and F.B. Weissler.

In order to prove the main result of the paper, we will assume (among other things) that A is globally bounded (together with its higher order derivatives). Clearly with this assumption the well-posedness and regularity features for (P) get easier to study. On the contrary, if A is unbounded, there are genuine regularity problems and the situation gets more involved [22].

Definition 2.1. *We say that a (sufficiently smooth) vector potential $A : \mathbb{R}^N \to \mathbb{R}^N$ is admissible with respect to problem (P) if the following Proposition 2.2 holds for A.*

Proposition 2.2. **[well-posedness statement]** *Assume that $0 < p < 2/N$. Then, for every $\varepsilon > 0$ and all $\phi_0 \in H_{A,\varepsilon}$, there exists a unique global solution*

$$\phi_\varepsilon \in C(\mathbb{R}^+, H_{A,\varepsilon}) \cap C^1(\mathbb{R}^+, H'_{A,\varepsilon})$$

of problem (P) with $\sup_{t\in\mathbb{R}^+} \|\phi_\varepsilon(t)\|_{H_{A,\varepsilon}} < \infty$. Moreover, the mass associated with $\phi_\varepsilon(t)$,

$$\mathcal{N}_\varepsilon(t) = \frac{1}{\varepsilon^N}\int |\phi_\varepsilon(t)|^2 dx,$$

as well as the total energy $E_\varepsilon(t) = \varepsilon^{-N}E(\phi_\varepsilon(t))$ associated with (P)

$$\begin{aligned} E_\varepsilon(t) = {} & \frac{1}{2\varepsilon^N}\int \left|\frac{\varepsilon}{\mathrm{i}}\nabla\phi_\varepsilon(t) - A(x)\phi_\varepsilon(t)\right|^2 dx \\ & + \frac{1}{\varepsilon^N}\int V(x)|\phi_\varepsilon(t)|^2 dx - \frac{1}{(p+1)\varepsilon^N}\int |\phi_\varepsilon(t)|^{2p+2} dx, \end{aligned} \tag{2.4}$$

are conserved in time, namely

$$\mathcal{N}_\varepsilon(t) = \mathcal{N}_\varepsilon(0), \quad E_\varepsilon(t) = E_\varepsilon(0), \qquad \textit{for all } t \in \mathbb{R}^+.$$

Finally if $\phi_0 \in H^2_{A,\varepsilon}$*, then* $\phi_\varepsilon \in C(\mathbb{R}^+, H^2_{A,\varepsilon}) \cap C^1(\mathbb{R}^+, L^2(\mathbb{R}^N;\mathbb{C}))$*.*

Remark 2.3. From Proposition 2.2, due to the choice of the initial data (I), the mass $\mathcal{N}_\varepsilon(t)$ is also *independent* of ε. Indeed, via the mass conservation and formula (2.2),

$$\mathcal{N}_\varepsilon(t) = \mathcal{N}_\varepsilon(0) = \frac{1}{\varepsilon^N}\int |\phi_\varepsilon(x,0)|^2 dx = \frac{1}{\varepsilon^N}\int \Big| r\Big(\frac{x-x_0}{\varepsilon}\Big)\Big|^2 dx = \|r\|^2_{L^2} = m, \tag{2.5}$$

for all $\varepsilon > 0$ and $t \in \mathbb{R}^+$.

2.3. **The driving Newtonian equation.** Given the initial data $x_0, \xi_0 \in \mathbb{R}^N$, we consider

$$x(t), \xi(t) : \mathbb{R}^+ \to \mathbb{R}^N,$$

being the (unique) global (under the regularity assumptions on V and A indicated below) solution of the first order differential system

$$\begin{cases} \dot{x}(t) = \xi(t), \\ \dot{\xi}(t) = -\nabla V(x(t)) - \xi(t) \times B(x(t)), \\ x(0) = x_0, \\ \xi(0) = \xi_0, \end{cases} \tag{2.6}$$

namely the second order ODE (1.4). Notice that, setting

$$\mathcal{H}(t) = \frac{1}{2}|\xi(t)|^2 + V(x(t)), \quad t \in \mathbb{R}^+, \tag{2.7}$$

$\mathcal{H}$ is a first integral associated with (2.6), namely

$$\mathcal{H}(t) = \mathcal{H}(0), \qquad \text{for all } t \in \mathbb{R}^+.$$

In general dimension N, this follows by the elementary observation that, as $\mathbb{H}^B(x)$ is a skew-symmetric matrix, we have $\xi(t) \cdot \mathbb{H}^B(x(t))\xi(t) = 0$ for all $t \in \mathbb{R}^+$.

2.4. **The main result.** We consider the following assumptions on the external electric and magnetic potentials, V and A.

(V) $V \in C^3(\mathbb{R}^N)$ is positive and $\|V\|_{C^3} < \infty$.
(A) $A \in C^3(\mathbb{R}^N;\mathbb{R}^N)$ with $\|A\|_{C^3} < \infty$ and A is admissible (cf. definition 2.1).

Consider $H^1(\mathbb{R}^N;\mathbb{C})$ equipped with the scaled norm $\|\phi\|_{\mathbb{H}_\varepsilon}$,

$$\|\phi\|^2_{\mathbb{H}_\varepsilon} = \varepsilon^{2-N}\|\nabla\phi\|^2_{L^2} + \varepsilon^{-N}\|\phi\|^2_{L^2}.$$

The main result of the paper is the following

Theorem 2.4. *Let* r *be the ground state solution of problem* (S) *and let* ϕ_ε *be the family of solutions to problem* (P) *with initial data* (I)*, for some* $x_0, \xi_0 \in \mathbb{R}^N$*. Let* $(x(t), \xi(t))$ *be the global solution to system* (2.6)*. Then there exist* $\delta > 0$ *and a locally uniformly bounded family of maps* $\theta_\varepsilon : \mathbb{R}^+ \to [0, 2\pi)$ *such that, if* $\|A\|_{C^2} < \delta$*, then*

$$\phi_\varepsilon(x,t) = r\Big(\frac{x - x(t)}{\varepsilon}\Big) e^{\frac{\mathrm{i}}{\varepsilon}[A(x(t))\cdot(x-x(t)) + x\cdot\xi(t) + \theta_\varepsilon(t)]} + \omega_\varepsilon, \tag{2.8}$$

locally uniformly in time, where $\omega_\varepsilon \in \mathbb{H}_\varepsilon$ *and* $\|\omega_\varepsilon\|_{\mathbb{H}_\varepsilon} = \mathcal{O}(\varepsilon)$*, as* $\varepsilon \to 0$*. Furthermore, without restrictions on* $\|A\|_{C^2}$*, we have*

$$|\phi_\varepsilon(x,t)| = r\Big(\frac{x - x(t)}{\varepsilon}\Big) + \hat{\omega}^j_\varepsilon(x,t), \tag{2.9}$$

locally uniformly in time, where $\hat{\omega}_\varepsilon \in \mathbb{H}_\varepsilon$ *and* $\|\hat{\omega}^j_\varepsilon\|_{\mathbb{H}_\varepsilon} \le \mathcal{O}(\varepsilon)$*, as* $\varepsilon \to 0$*.*

Some comments are now in order.

Remark 2.5. If x_0 is a *critical point* of V and $\xi_0 = 0$, then the solution of system (2.6) is $(x(t), \xi(t)) = (x_0, 0)$ for all $t \in \mathbb{R}^+$. Then, the conclusion of the previous result reads as

$$\phi_\varepsilon(x,t) = r\Big(\frac{x - x_0}{\varepsilon}\Big)e^{\frac{\mathrm{i}}{\varepsilon}[A(x_0)\cdot(x-x_0)+\theta_\varepsilon(t)]} + \omega_\varepsilon,$$

locally uniformly in time, where $\omega_\varepsilon \in \mathbb{H}_\varepsilon$ and $\|\omega_\varepsilon\|_{\mathbb{H}_\varepsilon} = \mathcal{O}(\varepsilon)$ as $\varepsilon \to 0$. In particular, this is consistent with the literature of the *standing wave* solutions of (P) in presence of a magnetic potential A (see e.g. [3, 7, 18, 19, 20, 21, 40] and references included).

Remark 2.6. In the framework of Theorem 2.4, by the exponential decay of r, it holds

$$|\phi_\varepsilon(x,t)| \le Ce^{-\sigma\frac{|x-x(t)|}{\varepsilon}} + |\omega_\varepsilon(x,t)|.$$

For an arbitrarily small $\delta > 0$, the solution ϕ_ε of (P) is expected to decay exponentially in the set $\mathcal{P}_\delta = \{x \in \mathbb{R}^N : |x - x(t)| \ge \delta > 0, \text{ for all } t \in \mathbb{R}^+\}$ faster and faster as $\varepsilon \to 0$, namely ϕ_ε rapidly vanishes far from the concentration curve $x(t)$.

Remark 2.7. A typical situation in $\mathbb{R}^3$ is when the external magnetic field $B = (b_1, b_2, b_3)$ is constant. Without loss of generality, up to a rotation, one can assume that $B = (0, 0, b)$ for some $b \in \mathbb{R}$. Hence, the corresponding vector potential is $A(x, y, z) = \frac{b}{2}(-y, x, 0)$. In this case, for harmonic external potentials V, namely

$$V(x_1, x_2, x_3) = \frac{1}{2}(\omega_1^2 x_1^2 + \omega_2^2 x_2^2 + \omega_3^2 x_3^2), \qquad \omega_j \in \mathbb{R},$$

system (2.6) reduces to

$$\begin{cases} \dot{x}_1(t) = \xi_1(t), \\ \dot{x}_2(t) = \xi_2(t), \\ \dot{x}_3(t) = \xi_3(t), \\ \dot{\xi}_1(t) = -\omega_1^2 x_1(t) - b\xi_2(t), \\ \dot{\xi}_2(t) = -\omega_2^2 x_2(t) + b\xi_1(t), \\ \dot{\xi}_3(t) = -\omega_3^2 x_3(t). \end{cases} \tag{2.10}$$

It is clear that, setting some fixed values of ω_j and choosing some initial data, enlarging the value of the third component b of the magnetic field B (say, from 0 to 60), the original periodic orbit turns into a more and more helicoidal pattern. See figures 1 and 2.

Remark 2.8. By complicating some arguments, assumption **(V)** could be relaxed. For instance V can be written as $V_1 + V_2$, being $\|V_1\|_{C^3} < \infty$ and V_2 sufficiently smooth. The idea is to use the cut-off function indicated in (3.5), which is nonzero in the ball of $\mathbb{R}^N$ containing the region where the orbit $x(t)$ is confined (see [39]).

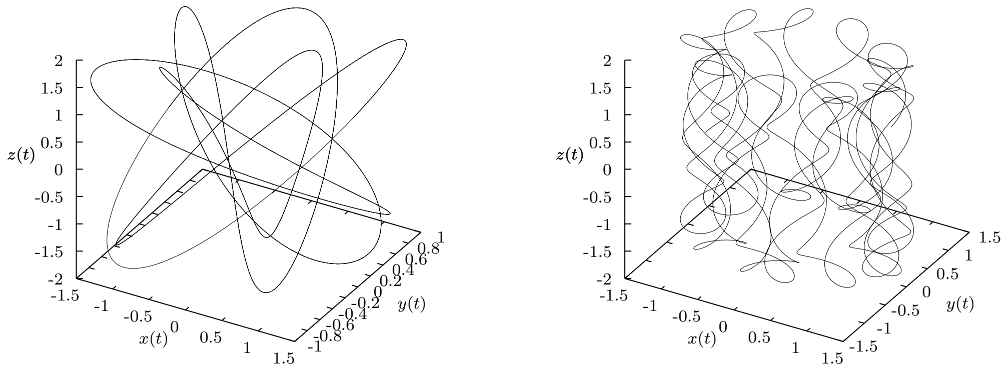

FIGURE 1. Phase portrait of system (2.10) with $\omega_1 = 1$, $\omega_2 = 1.4$, $\omega_3 = 1.2$, $b = 0$ (left, no magnetic field) and $b = 5$ (right, weak magnetic field).

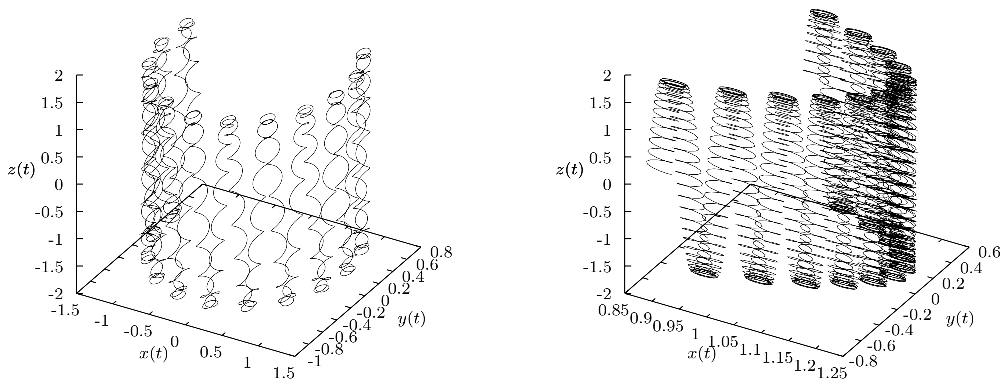

FIGURE 2. Phase portrait of system (2.10) with $\omega_1 = 1$, $\omega_2 = 1.4$, $\omega_3 = 1.2$, $b = 20$ (left) and $b = 60$ (right). The effects of the magnetic field increases.

Remark 2.9. As remarked in [39], the soliton dynamics behaviour *breaks down in the critical case* $p = \frac{2}{N}$. Indeed, in this case, if we choose $x_0 = \xi_0 = 0$, $V(x) = \frac{1}{2}|x|^2$ and $A = 0$, then the modulus of the solution of problem (P) with initial data $\phi_0(x) = r(\frac{x}{\varepsilon})$ is given by $|\phi_\varepsilon(x,t)| = (\cos t)^{-N/2} r(\frac{x}{\varepsilon \cos t})$ for all $x \in \mathbb{R}^N$ and $t \in [0, \frac{\pi}{2})$ (see also [13]).

3. Preliminary facts

In this section we collect some preliminary result which will allow us to prove the main result, Theorem 2.4.

3.1. **Magnetic momentum.** The following vector function is useful to pursue our goals.

Definition 3.1. *We define the momentum of the solution ϕ_ε, depending upon the vector potential A, as a function $p^A_\varepsilon : \mathbb{R}^N \times \mathbb{R}^+ \to \mathbb{R}^N$, by setting*

$$p^A_\varepsilon(x,t) := \frac{1}{\varepsilon^N} \Im\big(\bar{\phi}_\varepsilon(x,t)(\varepsilon \nabla \phi_\varepsilon(x,t) - \mathrm{i} A(x)\phi_\varepsilon(x,t)\big), \quad x \in \mathbb{R}^N,\ t \in \mathbb{R}^+.$$

First we state the following

Lemma 3.2. *Let ϕ_ε be the solution to problem (P) corresponding to the initial data (I). Then there exists a positive constant C such that*

$$\Big\|\frac{\varepsilon}{\mathrm{i}}\nabla\phi_\varepsilon(\cdot,t)-A(x)\phi_\varepsilon(\cdot,t)\Big\|_{L^2}^2\leq C\varepsilon^N,$$

for all $t\in\mathbb{R}^+$ and any $\varepsilon>0$.

Proof. The total energy $E_\varepsilon(t)$ is conserved (see Proposition 2.2) and it can be bounded independently of ε (see Lemma 3.5). Then, since V is positive, defining $\zeta_\varepsilon(x):=\phi_\varepsilon(\varepsilon x)$, it follows that, for some positive constant C,

$$\Big\|\frac{1}{\mathrm{i}}\nabla\zeta_\varepsilon(\cdot,t)-A(\varepsilon x)\zeta_\varepsilon(\cdot,t)\Big\|_{L^2}^2-C\|\zeta_\varepsilon(\cdot,t)\|_{L^{2p+2}}^{2p+2}\leq C. \tag{3.1}$$

By combining the diamagnetic inequality (see [23] for a proof)

$$|\nabla|\zeta_\varepsilon||\leq\Big|\Big(\frac{\nabla}{\mathrm{i}}-A(\varepsilon x)\Big)\zeta_\varepsilon\Big|,\qquad\text{a.e. in }\mathbb{R}^N$$

with the Gagliardo-Nirenberg inequality, setting $\vartheta=\frac{pN}{2p+2}\in(0,1)$, we obtain

$$\|\zeta_\varepsilon(\cdot,t)\|_{L^{2p+2}}\leq\|\zeta_\varepsilon(\cdot,t)\|_{L^2}^{1-\vartheta}\|\nabla|\zeta_\varepsilon(\cdot,t)|\|_{L^2}^{\vartheta}\leq\|\zeta_\varepsilon(\cdot,t)\|_{L^2}^{1-\vartheta}\Big\|\Big(\frac{\nabla}{\mathrm{i}}-A(\varepsilon x)\Big)\zeta_\varepsilon(\cdot,t)\Big\|_{L^2}^{\vartheta}.$$

By the conservation of mass (see Remark 2.3), we deduce that $\|\zeta_\varepsilon(\cdot,t)\|_{L^2}^2=\mathcal{N}_\varepsilon(t)=m$, independently of ε. Hence, for all $\varepsilon>0$, we get

$$\|\zeta_\varepsilon(\cdot,t)\|_{L^{2p+2}}^{2p+2}\leq C\Big\|\frac{1}{\mathrm{i}}\nabla\zeta_\varepsilon(\cdot,t)-A(\varepsilon x)\zeta_\varepsilon(\cdot,t)\Big\|_{L^2}^{pN}.$$

Since $pN<2$ by assumption, the assertion readily follows from (3.1) and rescaling. □

We have the following summability property for $p_\varepsilon^A(x,t)$.

Lemma 3.3. *There exists a positive constant C such that*

$$\sup_{t\in\mathbb{R}^+}\Big|\int p_\varepsilon^A(x,t)dx\Big|\leq C.$$

Proof. Taking into account the inequality of Lemma 3.2 and the mass conservation law, by Hölder inequality we get

$$\begin{aligned}\Big|\int p_\varepsilon^A(x,t)dx\Big|&\leq\int|p_\varepsilon^A(x,t)|dx\leq\frac{1}{\varepsilon^N}\int|\bar{\phi}_\varepsilon(x,t)|\Big|\frac{\varepsilon}{\mathrm{i}}\nabla\phi_\varepsilon(x,t)-A(x)\phi_\varepsilon(x,t)\Big|dx\\&\leq\frac{1}{\varepsilon^{N/2}}\|\phi_\varepsilon(\cdot,t)\|_{L^2}\frac{1}{\varepsilon^{N/2}}\Big\|\frac{\varepsilon}{\mathrm{i}}\nabla\phi_\varepsilon(\cdot,t)-A(x)\phi_\varepsilon(\cdot,t)\Big\|_{L^2}\leq C,\end{aligned}$$

for all $t\in\mathbb{R}^+$. The assertion follows by taking the supremum over positive times. □

3.2. **Energy levels in the semiclassical limit.** Let us recall a useful tool (see e.g. [39, Lemma 3.3]), which reveals useful in managing various estimates that follow.

Lemma 3.4. *Assume that $g:\mathbb{R}^N\to\mathbb{R}$ is a function of class $C^2(\mathbb{R}^N)$, $\|g\|_{C^2}<\infty$, and that r is the ground state solution of (S). Then, as ε goes to zero, it holds*

$$\int g(\varepsilon x+y)r^2(x)dx=\int g(y)r^2(x)dx+\mathcal{O}(\varepsilon^2),$$

for every $y \in \mathbb{R}^N$ *fixed. Moreover,* $\mathcal{O}(\varepsilon^2)$ *is uniform with respect to a family* $\mathcal{F} \subset C^2(\mathbb{R}^N)$ *which is uniformly bounded, that is* $\sup_{g\in\mathcal{F}} \|g\|_{C^2} < \infty$.

In the next lemma we compute the value of the energy associated with (P)-(I), in the semiclassical regime.

Lemma 3.5. *Let* E_ε *be the energy associated with the family* ϕ_ε *of solutions to problem* (P) *with initial data* (I). *Then, for every* $t \in \mathbb{R}^+$, *it holds*

$$E_\varepsilon(t) = \mathcal{E}(r) + m\mathcal{H}(t) + \mathcal{O}(\varepsilon^2),$$

as ε *goes to zero.*

Proof. Notice that, for all $x \in \mathbb{R}^N$, we get

$$\begin{aligned}&\Big(\frac{\varepsilon}{\mathrm{i}}\nabla - A(x)\Big)\Big(r\Big(\frac{x-x_0}{\varepsilon}\Big)e^{\frac{\mathrm{i}}{\varepsilon}[A(x_0)\cdot(x-x_0)+x\cdot\xi_0]}\Big) = \frac{1}{\mathrm{i}}e^{\frac{\mathrm{i}}{\varepsilon}[A(x_0)\cdot(x-x_0)+x\cdot\xi_0]}\nabla r\Big(\frac{x-x_0}{\varepsilon}\Big)\\&+ r\Big(\frac{x-x_0}{\varepsilon}\Big)e^{\frac{\mathrm{i}}{\varepsilon}[A(x_0)\cdot(x-x_0)+x\cdot\xi_0]}[A(x_0)+\xi_0] - r\Big(\frac{x-x_0}{\varepsilon}\Big)e^{\frac{\mathrm{i}}{\varepsilon}[A(x_0)\cdot(x-x_0)+x\cdot\xi_0]}A(x).\end{aligned}$$

Hence, it follows that

$$\begin{aligned}&\frac{1}{\varepsilon^N}\int\Big|\Big(\frac{\varepsilon}{\mathrm{i}}\nabla - A(x)\Big)\Big(r\Big(\frac{x-x_0}{\varepsilon}\Big)e^{\frac{\mathrm{i}}{\varepsilon}[A(x_0)\cdot(x-x_0)+x\cdot\xi_0]}\Big)\Big|^2dx = \frac{1}{\varepsilon^N}\int\Big|\nabla r\Big(\frac{x-x_0}{\varepsilon}\Big)\Big|^2dx\\&+\frac{1}{\varepsilon^N}\int r^2\Big(\frac{x-x_0}{\varepsilon}\Big)|A(x_0)+\xi_0|^2dx + \frac{1}{\varepsilon^N}\int r^2\Big(\frac{x-x_0}{\varepsilon}\Big)|A(x)|^2dx\\&-\frac{2}{\varepsilon^N}\int r^2\Big(\frac{x-x_0}{\varepsilon}\Big)A(x)\cdot(A(x_0)+\xi_0)dx\\&=\int|\nabla r(x)|^2\,dx + |A(x_0)+\xi_0|^2m + \int r^2(x)|A(\varepsilon x+x_0)|^2dx\\&-2\int r^2(x)A(\varepsilon x+x_0)\cdot(A(x_0)+\xi_0)dx.\end{aligned}$$

In view of Lemma 3.4, we have

$$\begin{aligned}&\int r^2(x)|A(\varepsilon x+x_0)|^2dx = |A(x_0)|^2m + \mathcal{O}(\varepsilon^2),\\&\int r^2(x)A(\varepsilon x+x_0)\cdot(A(x_0)+\xi_0)dx = A(x_0)\cdot(A(x_0)+\xi_0)m + \mathcal{O}(\varepsilon^2).\end{aligned}$$

Then,

$$\begin{aligned}&\frac{1}{\varepsilon^N}\int\Big|\Big(\frac{\varepsilon}{\mathrm{i}}\nabla - A(x)\Big)\Big(r\Big(\frac{x-x_0}{\varepsilon}\Big)e^{\frac{\mathrm{i}}{\varepsilon}[A(x_0)\cdot(x-x_0)+x\cdot\xi_0]}\Big)\Big|^2dx\\&=\int|\nabla r(x)|^2\,dx + |A(x_0)+\xi_0|^2m + |A(x_0)|^2m - 2A(x_0)\cdot(A(x_0)+\xi_0)m + \mathcal{O}(\varepsilon^2)\\&=\int|\nabla r(x)|^2\,dx + m|\xi_0|^2 + \mathcal{O}(\varepsilon^2).\end{aligned}$$

It turn, by combining the conservation of energy (see Proposition 2.2) and the conservation of the function $\mathcal{H}$ (see definition (2.7)), we get

$$
\begin{aligned}
E_\varepsilon(t) &= E_\varepsilon(0) = E_\varepsilon\Big(r\Big(\frac{x-x_0}{\varepsilon}\Big)e^{\frac{\mathrm{i}}{\varepsilon}[A(x_0)\cdot(x-x_0)+x\cdot\xi_0]}\Big) \\
&= \frac{1}{2\varepsilon^N}\int\Big|\Big(\frac{\varepsilon}{\mathrm{i}}\nabla - A(x)\Big)\Big(r\Big(\frac{x-x_0}{\varepsilon}\Big)e^{\frac{\mathrm{i}}{\varepsilon}[A(x_0)\cdot(x-x_0)+x\cdot\xi_0]}\Big)\Big|^2dx \\
&+ \int V(x_0+\varepsilon x)r^2(x)dx - \frac{1}{p+1}\int|r(x)|^{2p+2}dx \\
&= \frac{1}{2}\int|\nabla r(x)|^2\,dx + \int V(x_0+\varepsilon x)r^2(x)dx - \frac{1}{p+1}\int|r(x)|^{2p+2}dx + \frac{1}{2}m|\xi_0|^2 + \mathcal{O}(\varepsilon^2) \\
&= \mathcal{E}(r) + \int V(x_0+\varepsilon x)r^2(x)dx + \frac{1}{2}m|\xi_0|^2 + \mathcal{O}(\varepsilon^2) \\
&= \mathcal{E}(r) + mV(x_0) + \frac{1}{2}m|\xi_0|^2 + \mathcal{O}(\varepsilon^2) \\
&= \mathcal{E}(r) + m\mathcal{H}(0) + \mathcal{O}(\varepsilon^2) \\
&= \mathcal{E}(r) + m\mathcal{H}(t) + \mathcal{O}(\varepsilon^2),
\end{aligned}
$$

as ε goes to zero. □

Lemma 3.6. *Let ϕ_ε be the family of solutions to problem (P) with initial data (I). Let us set, for any $\varepsilon>0$, $t\in\mathbb{R}^+$ and $x\in\mathbb{R}^N$*

$$
\psi_\varepsilon(x,t) = e^{-\frac{\mathrm{i}}{\varepsilon}\xi(t)\cdot[\varepsilon x+x(t)]}e^{-\mathrm{i}A(x(t))\cdot x}\,\phi_\varepsilon(\varepsilon x+x(t),t) \tag{3.2}
$$

where $(x(t),\xi(t))$ is the solution of system (2.6). Then

$$
\begin{aligned}
\mathcal{E}(\psi_\varepsilon(t)) = E_\varepsilon(t) &- \int V(x)\frac{|\phi_\varepsilon(x,t)|^2}{\varepsilon^N}dx + \frac{1}{2}m|\xi(t)+A(x(t))|^2 \\
&- (\xi(t)+A(x(t))\cdot\int p_\varepsilon^A(x,t)dx - (\xi(t)+A(x(t))\cdot\int A(x)\frac{|\phi_\varepsilon(x,t)|^2}{\varepsilon^N}dx \\
&+ \frac{1}{2}\int|A(x)|^2\frac{|\phi_\varepsilon(x,t)|^2}{\varepsilon^N}dx + \int A(x)\cdot p_\varepsilon^A(x,t)dx.
\end{aligned}
$$

Proof. By a simple change of variable and Remark 2.3, we have

$$
\|\psi_\varepsilon(t)\|_{L^2}^2 = \|\phi_\varepsilon(\varepsilon x+x(t),t)\|_{L^2}^2 = \frac{1}{\varepsilon^N}\|\phi_\varepsilon(t)\|_{L^2}^2 = \mathcal{N}_\varepsilon(t) = m, \quad t\in\mathbb{R}^+. \tag{3.3}
$$

Hence the mass of $\psi_\varepsilon(t)$ is conserved during the evolution. Let

$$
p_\varepsilon(x,t) = \frac{1}{\varepsilon^{N-1}}\Im\big(\bar\phi_\varepsilon(x,t)\nabla\phi_\varepsilon(x,t)\big), \quad x\in\mathbb{R}^N,\ t\in\mathbb{R}^+,
$$

be the magnetic-free momentum. A direct computation yields

$$\begin{aligned}
&\mathcal{E}(\psi_\varepsilon(t)) = \frac{1}{2\varepsilon^{N-2}}\int |\nabla\phi_\varepsilon(t)|^2 dx + \frac{1}{2}m|\xi(t)+A(x(t))|^2\\
&-\frac{1}{\varepsilon^N}\frac{1}{p+1}\int |\phi_\varepsilon(t)|^{2p+2}dx - (\xi(t)+A(x(t))\cdot\int p_\varepsilon(x,t)dx\\
&=\frac{1}{2\varepsilon^N}\int \Big|\frac{\varepsilon}{\mathrm{i}}\nabla\phi_\varepsilon(t) - A(x)\phi_\varepsilon(t)\Big|^2 dx\\
&-\frac{1}{2\varepsilon^N}\int |A(x)|^2|\phi_\varepsilon(t)|^2dx + \frac{1}{\varepsilon^{N-1}}\int A(x)\cdot\Im(\bar{\phi}_\varepsilon(t)\nabla\phi_\varepsilon(t))\\
&+\frac{1}{2}m|\xi(t)+A(x(t))|^2 - \frac{1}{\varepsilon^N}\frac{1}{p+2}\int |\phi_\varepsilon(t)|^{2p+2}dx - (\xi(t)+A(x(t))\cdot\int p_\varepsilon(x,t)dx.
\end{aligned}$$

Then, taking into account the definition of $E_\varepsilon(t)$, we obtain

$$\begin{aligned}
\mathcal{E}(\psi_\varepsilon(t)) = E_\varepsilon(t) &- \int V(x)\frac{|\phi_\varepsilon(x,t)|^2}{\varepsilon^N}dx + \frac{1}{2}m|\xi(t)+A(x(t))|^2\\
&-(\xi(t)+A(x(t))\cdot\int p_\varepsilon(x,t)dx\\
&-\frac{1}{2\varepsilon^N}\int |A(x)|^2|\phi_\varepsilon(x,t)|^2dx + \int A(x)\cdot p_\varepsilon(x,t)dx.
\end{aligned}$$

Finally, since

$$p_\varepsilon(x,t) = p_\varepsilon^A(x,t) + \varepsilon^{-N}A(x)|\phi_\varepsilon(x,t)|^2,$$

we obtain the desired conclusion. □

Next we introduce two important functionals in the dual space of C^2.

Definition 3.7. *Let ϕ_ε be the family of solutions to problem (P) with initial data (I) and let p_ε^A be the corresponding momentum. For any $t\in\mathbb{R}^+$, let us define an element $\Pi_\varepsilon^1(\cdot,t)$ in the dual space of $C^2(\mathbb{R}^N;\mathbb{R}^N)$ and an element $\Pi_\varepsilon^2(\cdot,t)$ in the dual space of $C^2(\mathbb{R}^N;\mathbb{R})$ by setting*

$$\begin{aligned}
\forall\varphi\in C^2(\mathbb{R}^N;\mathbb{R}^N): \quad &\int \Pi_\varepsilon^1(x,t)\cdot\varphi\,dx = \int \varphi\cdot p_\varepsilon^A(x,t)dx - m\varphi(x(t))\cdot\xi(t),\\
\forall\varphi\in C^2(\mathbb{R}^N;\mathbb{R}): \quad &\int \Pi_\varepsilon^2(x,t)\varphi\,dx = \int \varphi\frac{|\phi_\varepsilon(x,t)|^2}{\varepsilon^N}dx - m\varphi(x(t)),
\end{aligned}$$

and all $t\in\mathbb{R}^+$. Here $x(t),\xi(t)$ denote the components of the solution of system (2.6).

We recall a property of the functional δ_y on $C^2(\mathbb{R}^N)$ (see [39, Lemma 3.1, 3.2]).

Lemma 3.8. *There exist three positive constants K_0, K_1, K_2 such that, for all $y,z\in\mathbb{R}^N$,*

$$K_1|y-z| \le \|\delta_y-\delta_z\|_{C^{2*}} \le K_2|y-z|,$$

provided that $\|\delta_y-\delta_z\|_{C^{2*}}\le K_0$.

For a fixed time $T_0>0$ (to be chosen later on), let ρ be a positive constant defined by

$$\rho = K_1\sup_{[0,T_0]}|x(t)| + K_0 \tag{3.4}$$

where $x(t)$ is defined in (2.6), the constants K_0 and K_1 are defined in Lemma 3.8, and let χ be a $C^\infty(\mathbb{R}^N)$ function such that $0 \le \chi \le 1$ and

$$\chi(x) = 1 \quad \text{if } |x| < \rho, \qquad \chi(x) = 0 \quad \text{if } |x| > 2\rho. \tag{3.5}$$

Let us now set, for all $t \in \mathbb{R}^+$ and $\varepsilon > 0$,

$$\begin{aligned}
\omega^1_\varepsilon(t) &:= \int (\xi(t) + A(x(t)) \cdot \Pi^1_\varepsilon(x,t)dx, \\
\omega^2_\varepsilon(t) &:= \int A(x) \cdot \Pi^1_\varepsilon(x,t)dx, \\
\omega^3_\varepsilon(t) &:= \int |A(x)|^2 \Pi^2_\varepsilon(x,t)dx, \\
\omega^4_\varepsilon(t) &:= \int (\xi(t) + A(x(t)) \cdot A(x)\Pi^2_\varepsilon(x,t)dx, \\
\omega^5_\varepsilon(t) &:= \int V(x)\Pi^2_\varepsilon(x,t)dx, \\
\gamma_\varepsilon(t) &:= mx(t) - \int x\chi(x)\frac{|\phi_\varepsilon(x,t)|^2}{\varepsilon^N}dx,
\end{aligned}$$

where χ is as in (3.5).

On the functions ω^j_ε, we have the following estimate.

Lemma 3.9. *There exists a positive constant $C = C(V, A)$ such that*

$$\sum_{j=1}^{5} |\omega^j_\varepsilon(t)| \le C\Omega_\varepsilon(t), \tag{3.6}$$

where the function $\Omega_\varepsilon : \mathbb{R}^+ \to \mathbb{R}^+$ is defined as $\Omega_\varepsilon(t) = \hat{\Omega}_\varepsilon(t) + \rho^A_\varepsilon(t)$, where

$$\hat{\Omega}_\varepsilon(t) := \Big|\int \Pi^1_\varepsilon(x,t)dx\Big| + \sup_{\|\varphi\|_{C^3}\le 1} \Big|\int \varphi\Pi^2_\varepsilon(x,t)dx\Big| + |\gamma_\varepsilon(t)|, \quad t \in \mathbb{R}^+, \tag{3.7}$$

$$\rho^A_\varepsilon(t) := \Big|\int A(x) \cdot \Pi^1_\varepsilon(x,t)dx\Big|, \quad t \in \mathbb{R}^+. \tag{3.8}$$

Moreover

$$\Omega_\varepsilon(0) = \mathcal{O}(\varepsilon^2),$$

as ε goes to zero.

Proof. Estimate (3.6) is a simple and direct consequence of the definition of $\omega^j_\varepsilon(t)$, $\Omega_\varepsilon(t)$, of the uniform boundedness of $\xi(t), A(x(t))$, namely $|\xi(t)| + |A(x(t))| \le C$ and of the fact that $\|V\|_{C^3} < \infty$ and $\|A\|_{C^3} < \infty$. Let us now prove that $\Omega_\varepsilon(0) = \mathcal{O}(\varepsilon^2)$, as $\varepsilon \to 0$. Recalling that the initial data ϕ_0 is $r((x - x_0)/\varepsilon)e^{i/\varepsilon[A(x_0)\cdot(x-x_0)+x\cdot\xi_0]}$, in light of Lemma 3.4, for any

$\varphi \in C^2(\mathbb{R}^N;\mathbb{R}^N)$ such that $\|\varphi\|_{C^2} \leq 1$, we infer

$$\begin{aligned}
&\int \varphi(x)\cdot \Pi^1_\varepsilon(x,0)dx = \int \varphi(x)\cdot p^A_\varepsilon(x,0)dx - m\varphi(x_0)\cdot\xi_0 \\
&= \frac{1}{\varepsilon^{N-1}}\int \varphi(x)\cdot \Im\big(\bar\phi_\varepsilon(x,0)\nabla\phi_\varepsilon(x,0)\big) \\
&- \frac{1}{\varepsilon^N}\int \varphi(x)\cdot A(x)|\phi_\varepsilon(x,0)|^2dx - m\varphi(x_0)\cdot\xi_0 \\
&= \frac{1}{\varepsilon^N}\int \varphi(x)\cdot (A(x_0)+\xi_0)r^2\left(\frac{x-x_0}{\varepsilon}\right)dx \\
&- \frac{1}{\varepsilon^N}\int \varphi(x)\cdot A(x)r^2\left(\frac{x-x_0}{\varepsilon}\right)dx - m\varphi(x_0)\cdot\xi_0 \\
&= \int \varphi(x_0+\varepsilon x)\cdot (A(x_0)+\xi_0)r^2(x)dx \\
&- \int \varphi(x_0+\varepsilon x)\cdot A(x_0+\varepsilon x)r^2(x)dx - m\varphi(x_0)\cdot\xi_0 \\
&= m\varphi(x_0)\cdot(A(x_0)+\xi_0) - m\varphi(x_0)\cdot A(x_0) \\
&- m\varphi(x_0)\cdot\xi_0 + \mathcal{O}(\varepsilon^2) = \mathcal{O}(\varepsilon^2),
\end{aligned}$$

as ε goes to zero. In a similar fashion, for any $\varphi \in C^3(\mathbb{R}^N)$ with $\|\varphi\|_{C^3} \leq 1$, we get

$$\begin{aligned}
\int \varphi(x)\Pi^2_\varepsilon(x,0)dx &= \frac{1}{\varepsilon^N}\int \varphi(x)|\phi_\varepsilon(x,0)|^2dx - m\varphi(x_0) \\
&= \int \varphi(x_0+\varepsilon x)r^2(x)dx - m\varphi(x_0) = \mathcal{O}(\varepsilon^2).
\end{aligned}$$

Finally, as $\chi(x_0)=1$, we have $|\gamma_\varepsilon(0)| = \big|mx_0 - \int(x_0+\varepsilon y)\chi(x_0+\varepsilon y)r^2(y)dy\big| \leq \mathcal{O}(\varepsilon^2)$, by Lemma 3.4. This concludes the proof of the assertion. $\square$

At this stage, we are ready to estimate the energy values $\mathcal{E}(\psi_\varepsilon(t))$.

Lemma 3.10. *Let ψ_ε be the function defined in formula (3.2). Then there exists a positive constant C such that*

$$0 \leq \mathcal{E}(\psi_\varepsilon(t)) - \mathcal{E}(r) \leq C\Omega_\varepsilon(t) + \mathcal{O}(\varepsilon^2),$$

for all $t \in \mathbb{R}^+$ and $\varepsilon > 0$.

Proof. By combining the conclusions of Lemma 3.5 and 3.6, we obtain

$$\begin{aligned}
\mathcal{E}(\psi_\varepsilon(t)) - \mathcal{E}(r) &= m\mathcal{H}(t) - \int V(x)\frac{|\phi_\varepsilon(x,t)|^2}{\varepsilon^N}dx + \frac{1}{2}m|\xi(t)+A(x(t))|^2 \\
&- (\xi(t)+A(x(t)))\cdot\int p^A_\varepsilon(x,t)dx - (\xi(t)+A(x(t))\cdot\int A(x)\frac{|\phi_\varepsilon(x,t)|^2}{\varepsilon^N}dx \\
&+ \frac{1}{2}\int |A(x)|^2\frac{|\phi_\varepsilon(x,t)|^2}{\varepsilon^N}dx + \int A(x)\cdot p^A_\varepsilon(x,t)dx + \mathcal{O}(\varepsilon^2),
\end{aligned}$$

for all $t \in \mathbb{R}^+$, as ε goes to zero. Notice that

$$(\xi(t) + A(x(t)) \cdot \int p_\varepsilon^A(x,t)dx = m|\xi(t)|^2 + mA(x(t)) \cdot \xi(t) + \omega_1(t),$$

$$\int A(x) \cdot p_\varepsilon^A(x,t)dx = mA(x(t)) \cdot \xi(t) + \omega_2(t),$$

$$\int |A(x)|^2 \frac{|\phi_\varepsilon(x,t)|^2}{\varepsilon^N} dx = m|A(x(t))|^2 + \omega_3(t),$$

$$\int (\xi(t) + A(x(t)) \cdot A(x) \frac{|\phi_\varepsilon(x,t)|^2}{\varepsilon^N} dx = m\xi(t) \cdot A(x(t)) + m|A(x(t))|^2 + \omega_4(t),$$

$$\int V(x) \frac{|\phi_\varepsilon(x,t)|^2}{\varepsilon^N} dx = mV(x(t)) + \omega_5(t).$$

It follows that

$$\begin{aligned}
\mathcal{E}(\psi_\varepsilon(t)) - \mathcal{E}(r) &= \frac{1}{2}m|\xi(t)|^2 + mV(x(t)) - mV(x(t)) - \omega_5(t) + \frac{1}{2}m|\xi(t) + A(x(t))|^2 \\
&\quad - m|\xi(t)|^2 - mA(x(t)) \cdot \xi(t) - \omega_1(t) - m\xi(t) \cdot A(x(t)) - m|A(x(t))|^2 - \omega_4(t) \\
&\quad + \frac{1}{2}m|A(x(t))|^2 + \frac{\omega_3(t)}{2} + mA(x(t)) \cdot \xi(t) + \omega_2(t) \\
&= -\omega_1(t) + \omega_2(t) + \frac{\omega_3(t)}{2} - \omega_4(t) - \omega_5(t) + \mathcal{O}(\varepsilon^2),
\end{aligned}$$

which concludes the proof in light of inequality (3.6) of Lemma 3.9. □

4. The approximation result

Let us first recall a useful and well-established stability property of ground states.

Proposition 4.1. *There exist two positive constants $\mathcal{A}$ and $\mathcal{C}$ such that, if $\Phi \in H^1(\mathbb{R}^N; \mathbb{C})$ is such that $\|\Phi\|_{L^2} = \|r\|_{L^2}$, where r is the ground state solution of (S), and*

$$\mathcal{E}(\Phi) - \mathcal{E}(r) \leq \mathcal{A},$$

then

$$\inf_{y \in \mathbb{R}^N,\, \vartheta \in [0,2\pi)} \|\Phi - e^{\mathrm{i}\theta} r(\cdot + y)\|_{H^1}^2 \leq \mathcal{C} \left(\mathcal{E}(\Phi) - \mathcal{E}(r) \right). \tag{4.1}$$

Proof. See [58, 59]. □

Next, in view of the previous preparatory work, we can state the reppresentation result.

Theorem 4.2. *Let ϕ_ε be the family of solutions to problem (P) with initial data (I) and let ψ_ε be the function defined in formula (3.2). Then there exist $\varepsilon_0 > 0$, a time $T_\varepsilon^* > 0$, families of uniformly bounded functions $\theta_\varepsilon : \mathbb{R}^+ \to [0, 2\pi)$, $y_\varepsilon : \mathbb{R}^+ \to \mathbb{R}^N$ and a positive constant C such that*

$$\phi_\varepsilon(x,t) = e^{\frac{\mathrm{i}}{\varepsilon}(\xi(t)\cdot x + \theta_\varepsilon(t) + A(x(t))\cdot(x - x(t))} r\Big(\frac{x - y_\varepsilon(t)}{\varepsilon}\Big) + \omega_\varepsilon(t), \tag{4.2}$$

where

$$\|\omega_\varepsilon(t)\|_{\mathbb{H}_\varepsilon} \leq C\sqrt{\Omega_\varepsilon(t)} + \mathcal{O}(\varepsilon),$$

for all $\varepsilon \in (0, \varepsilon_0)$ and $t \in [0, T_\varepsilon^*)$.

Proof. Since the function $\{t \mapsto \Omega_\varepsilon(t)\}$ defined in formula (3.6) is continuous, for any fixed $T_0 > 0$ and $\varepsilon_0, \sigma_0 > 0$, we can define the time (recall here that $\Omega(0) = \mathcal{O}(\varepsilon^2)$ as $\varepsilon \to 0$)

$$T^*_\varepsilon := \sup\big\{t \in [0, T_0] : \Omega_\varepsilon(s) \le \sigma_0, \text{ for all } s \in (0, t)\big\} > 0, \tag{4.3}$$

for all $\varepsilon \in (0, \varepsilon_0)$. Therefore, by choosing the numbers σ_0 and ε_0 sufficiently small, by virtue of Lemma 3.10, we conclude that

$$0 \le \mathcal{E}(\psi_\varepsilon(t)) - \mathcal{E}(r) \le C\Omega_\varepsilon(t) + \mathcal{O}(\varepsilon^2) \le \mathcal{A}, \quad \text{for all } \varepsilon \in (0, \varepsilon_0) \text{ and } t \in [0, T^*_\varepsilon).$$

Since $\|\psi_\varepsilon(t)\|_{L^2} = \|r\|_{L^2}$, we are in the right position to exploit the stability property of ground states (Proposition 4.1). Hence, there exist two families of uniformly bounded functions $\hat\theta_\varepsilon : \mathbb{R}^+ \to [0, 2\pi)$ and $\hat y_\varepsilon : \mathbb{R}^+ \to \mathbb{R}^N$ such that

$$\Big\|e^{-\frac{\mathrm{i}}{\varepsilon}\xi(t)\cdot[\varepsilon x + x(t)]}e^{-\mathrm{i}A(x(t))\cdot x}\,\phi_\varepsilon(\varepsilon x + x(t), t) - e^{\mathrm{i}\hat\theta_\varepsilon(t)}r\big(x + \hat y_\varepsilon(t)\big)\Big\|^2_{H^1} \le C\Omega_\varepsilon(t) + \mathcal{O}(\varepsilon^2),$$

for all $\varepsilon \in (0, \varepsilon_0)$ and any $t \in [0, T^*_\varepsilon)$. In turn, by rescaling and setting $\theta_\varepsilon(t) := \varepsilon\hat\theta_\varepsilon(t)$ and $y_\varepsilon(t) := x(t) - \varepsilon\hat y_\varepsilon(t)$, we get

$$\Big\|e^{-\frac{\mathrm{i}}{\varepsilon}\xi(t)\cdot x - \frac{\mathrm{i}}{\varepsilon}A(x(t))\cdot(x - x(t))}\phi_\varepsilon(x, t) - e^{\frac{\mathrm{i}}{\varepsilon}\theta_\varepsilon(t)}r\Big(\frac{x - y_\varepsilon(t)}{\varepsilon}\Big)\Big\|^2_{\mathbb{H}_\varepsilon} \le C\Omega_\varepsilon(t) + \mathcal{O}(\varepsilon^2),$$

for all $\varepsilon \in (0, \varepsilon_0)$ and $t \in [0, T^*_\varepsilon)$, namely inequality (4.2), concluding the proof. □

5. Mass and momentum identities

In the following lemma we obtain two important identities satisfied by the equation. Only after completion of the present paper, that the author discovered the second identity was independently obtained in [48]. For the sake of self-containedness we include our proof, which uses the first identity and it is shorter.

Lemma 5.1. *Let ϕ_ε be the solution to problem (P) corresponding to the initial data (I). Then we have the identity*

$$\frac{1}{\varepsilon^N}\frac{\partial|\phi_\varepsilon|^2}{\partial t}(x, t) = -\mathrm{div}_x\, p^A_\varepsilon(x, t), \quad x \in \mathbb{R}^N,\ t \in \mathbb{R}^+. \tag{5.1}$$

Moreover, for all $t \in \mathbb{R}^+$, we have the identity

$$\int \frac{\partial p^A_\varepsilon}{\partial t}(x, t)dx = -\int p^A_\varepsilon(x, t) \times B(x)dx - \int \nabla V(x)\frac{|\phi_\varepsilon(x, t)|^2}{\varepsilon^N}dx, \tag{5.2}$$

where $B = \nabla \times A$ is the magnetic field associated with the vector potential A.

Remark 5.2. The momentum identity (5.2), which plays an important rôle in our asymptotic analysis, can be thought as an extension of the so called *Ehrenfest's theorem* in presence of a magnetic field B.

Remark 5.3. It follows from the momentum identity (5.2) that for the nonlinear Schrödinger equation with no magnetic field ($\nabla \times A = 0$ in $\mathbb{R}^N$) and with a constant electric potential ($\nabla V = 0$ in $\mathbb{R}^N$) the momentum $t \mapsto \int p^A_\varepsilon(x, t)dx$ is a constant of motion.

Remark 5.4. Concerning the addenda in the right-end side of (5.2), in the semiclassical regime, by the upcoming Lemma 6.1, as $\varepsilon \to 0$,

$$\int p_\varepsilon^A(x,t) \times B(x)dx + \int \nabla V(x)\frac{|\phi_\varepsilon(x,t)|^2}{\varepsilon^N}dx \sim m\xi(t) \times B(y_\varepsilon(t)) + m\nabla V(y_\varepsilon(t)).$$

We will show that $y_\varepsilon(t)$ remains close to $x(t)$, for ε small (cf. Lemma 6.3). Hence, from the right-hand side of (5.2) the Newton equation (2.6) naturally emerges, ruling the dynamics of a particle subjected to an electric force $F_e = -\nabla V(x(t))$ and to a magnetic force $F_b = -v(t) \times B(x(t))$, being $v = \dot{x}$ the velocity.

Proof. By the exponential decay of $r(x), \partial_i r(x)$ and $\partial_{ij}^2 r(x)$ given by (2.3) and the fact that $\|A\|_{C^1} < \infty$, the initial data (I) belongs to $H^2_{A,\varepsilon}$. Hence, by the regularity (see Proposition 2.2), it follows that $\phi_\varepsilon(t)$ belongs to $H^1(\mathbb{R}^N;\mathbb{C}) \cap H^2_{A,\varepsilon}$ for all $t > 0$. By the standard Calderón-Zygmund inequality $\|\partial_{ij}^2\phi_\varepsilon(t)\|_{L^2} \leq C\|\Delta\phi_\varepsilon(t)\|_{L^2}$ for all t (see e.g. [29, Corollary 9.10]) and since, again, $\|A\|_{C^1} < \infty$, for any $i, j = 1, \dots, N$ we get

$$\begin{aligned}
\varepsilon^2\|\partial_{ij}^2\phi_\varepsilon(t)\|_{L^2} &\leq C\|\varepsilon^2\Delta\phi_\varepsilon(t)\|_{L^2} \leq C\big\|\big(\frac{\varepsilon}{\mathrm{i}}\nabla - A(x)\big)^2\phi_\varepsilon(t)\big\|_{L^2} + C\|A(x)\cdot\nabla\phi_\varepsilon(t)\|_{L^2} \\
&\quad + C\||A(x)|^2\phi_\varepsilon(t)\|_{L^2} + C\|\mathrm{div}_x A(x)\phi_\varepsilon(t)\|_{L^2} \\
&\quad \leq C\|\phi_\varepsilon(t)\|_{H^2_{A,\varepsilon}} + C\|\phi_\varepsilon(t)\|_{H^1} < \infty,
\end{aligned}$$

for all $t > 0$. Hence $\phi_\varepsilon(t) \in H^2(\mathbb{R}^N;\mathbb{C})$, for all $t > 0$. Set, for $j = 1, \dots, N$,

$$(p_\varepsilon^A)_j(x,t) = \frac{1}{\varepsilon^N}\Im\big(\bar{\phi}_\varepsilon(x,t)(\varepsilon\partial_j\phi_\varepsilon(x,t) - \mathrm{i}A_j(x)\phi_\varepsilon(x,t)\big).$$

To prove identity (5.1) notice that, on one hand, we have

$$\begin{aligned}
-\mathrm{div}_x\, p_\varepsilon^A(x,t) &= -\sum_{j=1}^N \partial_j(p_\varepsilon^A)_j(x,t) \\
&= -\sum_{j=1}^N \frac{1}{\varepsilon^N}\Im\big(\partial_j\bar{\phi}_\varepsilon(x,t)(\varepsilon\partial_j\phi_\varepsilon(x,t) - \mathrm{i}A_j(x)\phi_\varepsilon(x,t)\big) \\
&\quad - \sum_{j=1}^N \frac{1}{\varepsilon^N}\Im\big(\bar{\phi}_\varepsilon(x,t)(\varepsilon\partial_{jj}^2\phi_\varepsilon(x,t) - \mathrm{i}\partial_j A_j(x)\phi_\varepsilon(x,t) - \mathrm{i}A_j(x)\partial_j\phi_\varepsilon(x,t)\big) \\
&= \frac{2}{\varepsilon^N}A(x)\cdot\Re\big(\nabla\bar{\phi}_\varepsilon(x,t)\phi_\varepsilon(x,t)\big) \\
&\quad - \frac{1}{\varepsilon^{N-1}}\Im\big(\bar{\phi}_\varepsilon(x,t)\Delta\phi_\varepsilon(x,t)\big) + \frac{1}{\varepsilon^N}\mathrm{div}_x\, A(x)|\phi_\varepsilon(x,t)|^2.
\end{aligned}$$

On the other hand, it follows

$$
\begin{aligned}
\frac{1}{\varepsilon^N}\frac{\partial|\phi_\varepsilon|^2}{\partial t}(x,t) &= \frac{2}{\varepsilon^{N+1}}\Im\big(\bar{\phi}_\varepsilon(x,t)\big[\frac{1}{2}\big(\frac{\varepsilon}{\mathrm{i}}\nabla - A(x)\big)^2\phi_\varepsilon(x,t) + V(x)\phi_\varepsilon(x,t) - |\phi_\varepsilon(x,t)|^{2p}\phi_\varepsilon(x,t)\big]\big) \\
&= \frac{1}{\varepsilon^{N+1}}\Im\big(\bar{\phi}_\varepsilon(x,t)\big(\frac{\varepsilon}{\mathrm{i}}\nabla - A(x)\big)^2\phi_\varepsilon(x,t)\big) \\
&= -\frac{1}{\varepsilon^{N-1}}\Im\big(\bar{\phi}_\varepsilon(x,t)\Delta\phi_\varepsilon(x,t)) + \frac{2}{\varepsilon^N}A(x)\cdot\Re\big(\phi_\varepsilon(x,t)\nabla\bar{\phi}_\varepsilon(x,t)\big) \\
&+ \frac{1}{\varepsilon^N}\mathrm{div}_x\, A(x)|\phi_\varepsilon(x,t)|^2.
\end{aligned}
$$

Now, concerning second identity, (5.2), for any $j = 1,\dots,N$, it holds

$$
\begin{aligned}
\frac{\partial(p_\varepsilon^A)_j}{\partial t} &= \varepsilon^{1-N}\Im(\partial_t\overline{\phi}_\varepsilon\partial_j\phi_\varepsilon) + \varepsilon^{1-N}\Im(\overline{\phi}_\varepsilon\partial_j(\partial_t\phi_\varepsilon)) - \frac{1}{\varepsilon^N}A_j(x)\frac{\partial|\phi_\varepsilon|^2}{\partial t} \\
&= \varepsilon^{1-N}\Im(\partial_t\overline{\phi}_\varepsilon\partial_j\phi_\varepsilon) + \varepsilon^{1-N}\Im(\partial_j\big(\overline{\phi}_\varepsilon\partial_t\phi_\varepsilon\big)) - \varepsilon^{1-N}\Im(\partial_j\overline{\phi}_\varepsilon\partial_t\phi_\varepsilon) - \frac{1}{\varepsilon^N}A_j(x)\frac{\partial|\phi_\varepsilon|^2}{\partial t} \\
&= 2\varepsilon^{1-N}\Im(\partial_t\overline{\phi}_\varepsilon\partial_j\phi_\varepsilon) + \varepsilon^{1-N}\Im(\partial_j\big(\overline{\phi}_\varepsilon\partial_t\phi_\varepsilon\big)) - \frac{1}{\varepsilon^N}A_j(x)\frac{\partial|\phi_\varepsilon|^2}{\partial t}.
\end{aligned}
$$

The second term integrates to zero. Moreover, taking into account identity (5.1), we get

$$
\begin{aligned}
-\int\frac{1}{\varepsilon^N}A_j(x)\frac{\partial|\phi_\varepsilon|^2}{\partial t}(x,t)dx &= \int A_j(x)\mathrm{div}_x\, p_\varepsilon^A(x,t)dx = -\int\nabla A_j(x)\cdot p_\varepsilon^A(x,t)dx \\
&= -\varepsilon^{1-N}\int\sum_{i=1}^N\partial_i A_j(x)\Im\big(\bar{\phi}_\varepsilon(x,t)\partial_i\phi_\varepsilon(x,t)\big)dx \\
&+ \varepsilon^{-N}\int\sum_{i=1}^N A_i(x)\partial_i A_j(x)|\phi_\varepsilon(x,t)|^2dx.
\end{aligned}
$$

Concerning the first term in the formula for $\partial_t(p_\varepsilon^A)_j$, conjugate the equation, multiply it by $2\varepsilon^{-N}\partial_j\phi_\varepsilon$ and take the imaginary part. It follows (summation on repeated i indexes)

$$
\begin{aligned}
2\varepsilon^{1-N}\Im(\partial_t\overline{\phi_\varepsilon}\partial_j\phi_\varepsilon) &= -\varepsilon^{2-N}\Re(\Delta\overline{\phi_\varepsilon}\partial_j\phi_\varepsilon) + \varepsilon^{-N}|A(x)|^2\Re(\overline{\phi_\varepsilon}\partial_j\phi_\varepsilon) \\
&+ \varepsilon^{1-N}\mathrm{div}_x A(x)\Im(\bar{\phi}_\varepsilon\partial_j\phi_\varepsilon) + 2\varepsilon^{1-N}A(x)\cdot\Im(\nabla\bar{\phi}_\varepsilon\partial_j\phi_\varepsilon) \\
&+ 2\varepsilon^{-N}V(x)\Re(\overline{\phi_\varepsilon}\partial_j\phi_\varepsilon) - 2\varepsilon^{-N}|\phi_\varepsilon|^{2p}\Re(\overline{\phi_\varepsilon}\partial_j\phi_\varepsilon) \\
&= -\varepsilon^{2-N}\Re(\partial_i\big(\partial_i\overline{\phi_\varepsilon}\partial_j\phi_\varepsilon)) + \varepsilon^{2-N}\partial_j\Big(\frac{|\partial_i\phi_\varepsilon|^2}{2}\Big) \\
&+ \varepsilon^{-N}|A(x)|^2\Re(\overline{\phi_\varepsilon}\partial_j\phi_\varepsilon) + \varepsilon^{1-N}\mathrm{div}_x A(x)\Im(\bar{\phi}_\varepsilon\partial_j\phi_\varepsilon) \\
&+ 2\varepsilon^{1-N}A(x)\cdot\Im(\nabla\bar{\phi}_\varepsilon\partial_j\phi_\varepsilon) + \varepsilon^{-N}\partial_j\left(V(x)|\phi_\varepsilon|^2\right) \\
&- \varepsilon^{-N}\partial_j V(x)|\phi_\varepsilon|^2 - \varepsilon^{-N}\frac{1}{p+1}\partial_j\big(|\phi_\varepsilon|^{2p+2}\big).
\end{aligned}
$$

Notice that the following identity can be easily shown (recall that $\phi_\varepsilon(t) \in H^2$ for all t),

$$\int \mathrm{div}_x A(x)\Im(\bar\phi_\varepsilon \partial_j \phi_\varepsilon)dx + 2\int A(x)\cdot \Im(\nabla\bar\phi_\varepsilon \partial_j \phi_\varepsilon)dx = \int \sum_{i=1}^N \partial_j A_i(x)\Im\big(\bar\phi_\varepsilon \partial_i \phi_\varepsilon\big)dx.$$

Then, recalling that $\mathbb{H}^B = (\partial_j A_i - \partial_i A_j)_{ij}$ and that the flux of ϕ_ε is in H^2, we infer that

$$\begin{aligned}
\int \frac{\partial(p^A_\varepsilon)_j}{\partial t} &= -\varepsilon^{-N}\int \sum_{i=1}^N A_i(x)\big(\partial_j A_i(x) - \partial_i A_j(x)\big)|\phi_\varepsilon|^2 dx \\
&\quad + \varepsilon^{1-N}\int \sum_{i=1}^N \partial_j A_i(x)\cdot \Im(\bar\phi_\varepsilon \partial_i\phi_\varepsilon)dx - \varepsilon^{-N}\int \partial_j V(x)|\phi_\varepsilon|^2 dx \\
&\quad - \varepsilon^{1-N}\int \sum_{i=1}^N \partial_i A_j(x)\cdot \Im\big(\bar\phi_\varepsilon \partial_i\phi_\varepsilon\big)dx \\
&= -\varepsilon^{-N}\int \sum_{i=1}^N A_i(x)\big(\partial_j A_i(x) - \partial_i A_j(x)\big)|\phi_\varepsilon|^2 dx \\
&\quad + \varepsilon^{1-N}\int \sum_{i=1}^N (\partial_j A_i(x) - \partial_i A_j(x))\cdot \Im(\bar\phi_\varepsilon \partial_i\phi_\varepsilon)dx - \varepsilon^{-N}\int \partial_j V(x)|\phi_\varepsilon|^2 dx \\
&= \int (\mathbb{H}^B p^A_\varepsilon(x,t))_j dx - \varepsilon^{-N}\int \partial_j V(x)|\phi_\varepsilon|^2 dx.
\end{aligned}$$

Taking into account the formal identification of the notation $-p^A_\varepsilon(x,t)\times B(x)$ with the matrix operation $\mathbb{H}^B p^A_\varepsilon(x,t)$, we obtain the assertion. To see this in the three dimensional case, recalling that

$$(B_1, B_2, B_3) = \nabla \times A = (\partial_2 A_3 - \partial_3 A_2, \partial_3 A_1 - \partial_1 A_3, \partial_1 A_2 - \partial_2 A_1),$$

we obtain the skew-symmetric matrix

$$\mathbb{H}^B(x) = \begin{bmatrix} 0 & \partial_2 A_1 - \partial_1 A_2 & \partial_3 A_1 - \partial_1 A_3 \\ \partial_1 A_2 - \partial_2 A_1 & 0 & \partial_3 A_2 - \partial_2 A_3 \\ \partial_1 A_3 - \partial_3 A_1 & \partial_2 A_3 - \partial_3 A_2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -B_3 & B_2 \\ B_3 & 0 & -B_1 \\ -B_2 & B_1 & 0 \end{bmatrix}.$$

Then, setting $p^i_\varepsilon = (p^A_\varepsilon)_i$, it follows that

$$\mathbb{H}^B(x)p^A_\varepsilon(x,t) = \begin{bmatrix} 0 & -B_3 & B_2 \\ B_3 & 0 & -B_1 \\ -B_2 & B_1 & 0 \end{bmatrix}\begin{bmatrix} p^1_\varepsilon \\ p^2_\varepsilon \\ p^3_\varepsilon \end{bmatrix} = \begin{bmatrix} p^3_\varepsilon B_2 - p^2_\varepsilon B_3 \\ p^1_\varepsilon B_3 - p^3_\varepsilon B_1 \\ p^2_\varepsilon B_1 - p^1_\varepsilon B_2 \end{bmatrix} = -p^A_\varepsilon(x,t)\times B(x).$$

The proof is now concluded. □

6. Mass and momentum estimates

First, we have the following control on the mass and momentum.

Lemma 6.1. *Let $\varepsilon_0 > 0$, $T^*_\varepsilon > 0$ and $y_\varepsilon(t)$ be as in Theorem 4.2. Then there exists a positive constant C such that*

$$\Big\|\frac{|\phi_\varepsilon(x,t)|^2}{\varepsilon^N}dx - m\delta_{y_\varepsilon(t)}\Big\|_{(C^2)^*} + \Big\|p_\varepsilon^{A(x(t))}(x,t)dx - m\xi(t)\delta_{y_\varepsilon(t)}\Big\|_{(C^2)^*} \leq C\Omega_\varepsilon(t) + \mathcal{O}(\varepsilon^2),$$

for every $t \in [0, T^*_\varepsilon)$ and $\varepsilon \in (0, \varepsilon_0)$.

Proof. For any $v \in H^1(\mathbb{R}^N;\mathbb{C})$, we have $|\nabla|v||^2 = |\nabla v|^2 - \frac{|\Im(\bar{v}\nabla v)|^2}{|v|^2}$. Then, if $\psi_\varepsilon(x,t)$ is the function introduced in formula (3.2), by Lemma 3.10 it follows that

$$0 \leq \mathcal{E}(|\psi_\varepsilon|) - \mathcal{E}(r) + \frac{1}{2}\int \frac{|\Im(\bar{\psi}_\varepsilon \nabla\psi_\varepsilon)|^2}{|\psi_\varepsilon|^2}dx \leq C\Omega_\varepsilon(t) + \mathcal{O}(\varepsilon^2),$$

for every $t \in [0, T^*_\varepsilon)$ and $\varepsilon \in (0, \varepsilon_0)$. Moreover, as $\||\psi_\varepsilon|\|_{L^2} = \|r\|_{L^2}$, by (2.1) we have

$$\int \frac{|\Im(\bar{\psi}_\varepsilon \nabla\psi_\varepsilon)|^2}{|\psi_\varepsilon|^2}dx \leq C\Omega_\varepsilon(t) + \mathcal{O}(\varepsilon^2), \tag{6.1}$$

for every $t \in [0, T^*_\varepsilon)$ and $\varepsilon \in (0, \varepsilon_0)$. Now, by the definition of ψ_ε (cf. (3.2)), we get

$$\begin{aligned}
&\frac{|\Im(\bar{\psi}_\varepsilon \nabla\psi_\varepsilon)|^2}{|\psi_\varepsilon|^2} \\
&= \frac{\big|\Im(\bar{\phi}_\varepsilon(\varepsilon x + x(t), t)\varepsilon\nabla\phi_\varepsilon(\varepsilon x + x(t), t)) - (\xi(t) + A(x(t))|\phi_\varepsilon(\varepsilon x + x(t), t)|^2\big|^2}{|\phi_\varepsilon(\varepsilon x + x(t), t)|^2} \\
&= \frac{\big|\varepsilon^N p_\varepsilon^{A(x(t))}(\varepsilon x + x(t), t) - \xi(t)|\phi_\varepsilon(\varepsilon x + x(t), t)|^2\big|^2}{|\phi_\varepsilon(\varepsilon x + x(t), t)|^2} \\
&= \varepsilon^{2N}\frac{\big|p_\varepsilon^{A(x(t))}(\varepsilon x + x(t), t)\big|^2}{|\phi_\varepsilon(\varepsilon x + x(t), t)|^2} + |\xi(t)|^2|\phi_\varepsilon(\varepsilon x + x(t), t)|^2 \\
&- 2\varepsilon^N\xi(t)\cdot p_\varepsilon^{A(x(t))}(\varepsilon x + x(t), t).
\end{aligned}$$

Hence, by a change of variable, we reach

$$\int \frac{|\Im(\bar{\psi}_\varepsilon \nabla\psi_\varepsilon)|^2}{|\psi_\varepsilon|^2}dx = \varepsilon^N\int \frac{\big|p_\varepsilon^{A(x(t))}(x,t)\big|^2}{|\phi_\varepsilon(x,t)|^2}dx + m|\xi(t)|^2 - 2\xi(t)\cdot\int p_\varepsilon^{A(x(t))}(x,t)dx. \tag{6.2}$$

Notice that by simple computations, by combining (6.1) and (6.2), it holds

$$\begin{aligned}
\int \Big|\varepsilon^{N/2}\frac{p_\varepsilon^{A(x(t))}(x,t)}{|\phi_\varepsilon(x,t)|} - \frac{\int p_\varepsilon^{A(x(t))}(x,t)dx}{m}\frac{|\phi_\varepsilon(x,t)|}{\varepsilon^{N/2}}\Big|^2 + m\Big|\xi(t) - \frac{\int p_\varepsilon^{A(x(t))}(x,t)dx}{m}\Big|^2 \\
\leq C\Omega_\varepsilon(t) + \mathcal{O}(\varepsilon^2),
\end{aligned} \tag{6.3}$$

for every $t \in [0, T^*_\varepsilon)$ and $\varepsilon \in (0, \varepsilon_0)$. To prove the assertion, we estimate $\rho_\varepsilon(t)$, where

$$\rho_\varepsilon(t) := \Big|\int \psi(x)\frac{|\phi_\varepsilon(x,t)|^2}{\varepsilon^N}dx - m\psi(y_\varepsilon)\Big| + \Big|\int p_\varepsilon^{A(x(t))}(x,t)\psi(x) - m\xi(t)\psi(y_\varepsilon)\Big| \tag{6.4}$$

for every function ψ of class C^2 such that $\|\psi\|_{C^2} \leq 1$. Taking into account that, by the definition of $\Omega_\varepsilon(t)$ (cf. formula (3.7)) and $\|A - A(x(t))\|_{C^3} \leq C$, we have

$$\begin{aligned}
\Big|\int p_\varepsilon^{A(x(t))}(x,t)dx - m\xi(t)\Big| &\leq \Big|\int p_\varepsilon^{A}(x,t)dx - m\xi(t)\Big| + \Big|\int (p_\varepsilon^{A}(x,t) - p_\varepsilon^{A(x(t))}(x,t))dx\Big| \\
&\leq C\Omega_\varepsilon(t) + \Big|\int (A(x) - A(x(t)))\frac{|\phi_\varepsilon(x,t)|^2}{\varepsilon^N}dx\Big| \\
&= C\Omega_\varepsilon(t) + \Big|\int (A(x) - A(x(t)))\Big(\frac{|\phi_\varepsilon(x,t)|^2}{\varepsilon^N} - m\delta_{x(t)}\Big)dx\Big| \\
&\leq C\Omega_\varepsilon(t),
\end{aligned}$$

we can conclude that

$$\begin{aligned}
&\Big|\int p_\varepsilon^{A(x(t))}(x,t)\psi(x)dx - m\xi(t)\psi(y_\varepsilon(t))\Big| \leq \\
&\leq \Big|\int p_\varepsilon^{A(x(t))}(x,t)[\psi(x) - \psi(y_\varepsilon(t))]dx\Big| + |\psi(y_\varepsilon(t))|\Big|\int p_\varepsilon^{A(x(t))}(x,t)dx - m\xi(t)\Big| \\
&\leq \Big|\int p_\varepsilon^{A(x(t))}(x,t)[\psi(x) - \psi(y_\varepsilon(t))]dx\Big| + C\Omega_\varepsilon(t) \\
&\leq \frac{1}{m}\Big|\int p_\varepsilon^{A(x(t))}(x,t)dx\Big|\Big|\int \frac{\psi(x)|\phi_\varepsilon(x,t)|^2}{\varepsilon^N}dx - m\psi(y_\varepsilon(t))\Big| \\
&\quad + \Big|\int \psi(x)\Big[p_\varepsilon^{A(x(t))}(x,t) - \frac{1}{m}\Big(\int p_\varepsilon^{A(x(t))}(x,t)dx\Big)\frac{|\phi_\varepsilon(x,t)|^2}{\varepsilon^N}\Big]dx\Big| + C\Omega_\varepsilon(t),
\end{aligned}$$

for all $\varepsilon \in (0,\varepsilon_0)$ and $t \in [0,T_\varepsilon^*)$. Since $\int p_\varepsilon^{A(x(t))}(x,t)dx$ is bounded (see Lemma 3.3) and

$$\int \Big[p_\varepsilon^{A(x(t))}(x,t) - \frac{1}{m}\Big(\int p_\varepsilon^{A(x(t))}(x,t)dx\Big)\frac{|\phi_\varepsilon(x,t)|^2}{\varepsilon^N}\Big]dx = 0,$$

setting $\hat\psi(x) := \psi(x) - \psi(y_\varepsilon(t))$, it holds

$$\begin{aligned}
\rho_\varepsilon(t) &\leq \int |\hat\psi(x)|\frac{|\phi_\varepsilon(x,t)|^2}{\varepsilon^N}dx + C\int |\hat\psi(x)|\frac{|\phi_\varepsilon(x,t)|^2}{\varepsilon^N}dx \\
&\quad + \int |\hat\psi(x)|\Big|p_\varepsilon^{A(x(t))}(x,t) - \frac{1}{m}\Big(\int p_\varepsilon^{A(x(t))}(x,t)dx\Big)\frac{|\phi_\varepsilon(x,t)|^2}{\varepsilon^N}\Big|dx + C\Omega_\varepsilon(t).
\end{aligned}$$

From Young inequality and estimate (6.3), it follows

$$\begin{aligned}
\rho_\varepsilon(t) &\leq \int \big[C|\hat\psi(x)| + \frac{1}{2}|\hat\psi(x)|^2\big]\frac{|\phi_\varepsilon(x,t)|^2}{\varepsilon^N}dx \\
&\quad + \frac{1}{2}\int \Big|\varepsilon^{N/2}\frac{p_\varepsilon^{A(x(t))}(x,t)}{|\phi_\varepsilon(x,t)|} - \frac{1}{m}\Big(\int p_\varepsilon^{A(x(t))}(x,t)dx\Big)\frac{|\phi_\varepsilon(x,t)|}{\varepsilon^{N/2}}\Big|^2 + C\Omega_\varepsilon(t) \\
&\leq \int \big[C|\hat\psi(x)| + \frac{1}{2}|\hat\psi(x)|^2\big]\frac{|\phi_\varepsilon(x,t)|^2}{\varepsilon^N}dx + C\Omega_\varepsilon(t) + \mathcal{O}(\varepsilon^2).
\end{aligned} \tag{6.5}$$

Via inequality $a^2 \leq 2b^2 + 2(a-b)^2$ with $a = \varepsilon^{-N/2}|\phi_\varepsilon(x,t)|$ and $b = \varepsilon^{-N/2} r((x - y_\varepsilon(t))/\varepsilon)$,

$$\rho_\varepsilon(t) \leq \frac{C}{\varepsilon^N}\int \big[|\hat\psi(x)| + |\hat\psi(x)|^2\big] r^2\Big(\frac{x - y_\varepsilon(t)}{\varepsilon}\Big)dx + \frac{C}{\varepsilon^N}\int \Big||\phi_\varepsilon(x,t)| - r\Big(\frac{x - y_\varepsilon(t)}{\varepsilon}\Big)\Big|^2 dx$$
$$+ C\Omega_\varepsilon(t) + \mathcal{O}(\varepsilon^2) \leq \Omega_\varepsilon(t) + \mathcal{O}(\varepsilon^2),$$

for all $\varepsilon \in (0, \varepsilon_0)$ and $t \in [0, T^*_\varepsilon)$, by Lemma 3.4 (as $\hat\psi(y_\varepsilon(t)) = 0$) and Theorem 4.2. □

Next, we need to show that the distance between the points $y_\varepsilon(t)$ found out in the proof of Theorem 4.2 and the trajectory $x(t)$ is controlled by $\Omega_\varepsilon(t)$, as ε goes to zero.

Remark 6.2. We stress that in the proof of the next Lemma we will choose the value of T_0 that was introduced in formula (4.3) inside the definition of T^*_ε.

Lemma 6.3. *Let $y_\varepsilon(t)$ be as in Theorem 4.2. There exist positive constants ε_0, σ_0 and T_0, namely the values introduced in (4.3) in the definition of T^*_ε such that, for some positive constant C,*

$$|x(t) - y_\varepsilon(t)| \leq C\Omega_\varepsilon(t) + \mathcal{O}(\varepsilon^2),$$

for all $t \in [0, T^*_\varepsilon)$ and $\varepsilon \in (0, \varepsilon_0)$.

Proof. We first show that there exists a time T_0 such that $|y_\varepsilon(t)| < \rho$, for every $t \in [0, T^*_\varepsilon)$ with $T^*_\varepsilon \leq T_0$, where ρ is the positive constant introduced in formula (3.4). Let us first prove that $\|\delta_{y_\varepsilon(t_2)} - \delta_{y_\varepsilon(t_1)}\|_{C^{2*}} < \rho$ for all t_1, $t_2 \in [0, T^*_\varepsilon)$. Let $\varphi \in C^2(\mathbb{R}^N)$ be such that $\|\varphi\|_{C^2} \leq 1$. Hence, taking into account Lemma 3.3 and identity (5.1), we get

$$\int \Big(\frac{|\phi_\varepsilon(x,t_2)|^2}{\varepsilon^N} - \frac{|\phi_\varepsilon(x,t_1)|^2}{\varepsilon^N}\Big)\varphi(x)dx = \int\int_{t_1}^{t_2} \frac{1}{\varepsilon^N}\frac{\partial|\phi_\varepsilon|^2}{\partial t}(x,t)\varphi(x)dtdx$$
$$= \int\int_{t_1}^{t_2} -\varphi(x)\mathrm{div}_x\, p^A_\varepsilon(x,t)dtdx = \int_{t_1}^{t_2}\int \nabla\varphi(x)\cdot p^A_\varepsilon(x,t)dxdt$$
$$\leq \|\nabla\varphi\|_{L^\infty}\int_{t_1}^{t_2} dt\int |p^A_\varepsilon(x,t)|dx \leq C\|\varphi\|_{C^2}|t_2 - t_1| \leq C|t_2 - t_1|.$$

Hence, for all t_1, $t_2 \in [0, T^*_\varepsilon)$, it holds

$$\Big\|\frac{|\phi_\varepsilon(x,t_2)|^2}{\varepsilon^N}dx - \frac{|\phi_\varepsilon(x,t_1)|^2}{\varepsilon^N}dx\Big\|_{C^{2*}} \leq C|t_2 - t_1|.$$

In view of Lemma 6.1, the following inequality holds,

$$m\|\delta_{y_\varepsilon(t_2)} - \delta_{y_\varepsilon(t_1)}\|_{C^{2*}} \leq CT_0 + C\Omega_\varepsilon(t) + \mathcal{O}(\varepsilon^2) \leq C(T_0 + \sigma_0) + \mathcal{O}(\varepsilon^2).$$

Here we choose the value of T_0 and then of σ_0, ε_0 so small that

$$C(T_0 + \sigma_0) + \mathcal{O}(\varepsilon^2) < \min\{mK_0, mK_0K_1\},$$

being K_0 and K_1 the constants introduced in Lemma 3.8. Hence, $|y_\varepsilon(t_2) - y_\varepsilon(t_1)| < K_0$ for all t_1, $t_2 \in [0, T^*_\varepsilon)$, and since $y_\varepsilon(0) = x_0$, we obtain the desired assertion. We can now conclude the proof of this Lemma. The properties of the function χ imply

$$|x(t) - y_\varepsilon(t)| \leq \frac{1}{m}|\gamma_\varepsilon(t)| + \frac{1}{m}\Big|\int x\chi(x)\frac{|\phi_\varepsilon(x,t)|^2}{\varepsilon^N}dx - my_\varepsilon(t)\Big|.$$

In light of the first step of the proof, we have $\chi(y_\varepsilon(t)) = 1$ for all $t \in [0, T^*_\varepsilon)$ and $\varepsilon \in (0, \varepsilon_0)$, so that exploiting again Lemma 6.1, we conclude that

$$|x(t) - y_\varepsilon(t)| \le C\Omega_\varepsilon(t) + C\|x\chi\|_{C^2}\Big\|\frac{|\phi_\varepsilon(x,t)|^2}{\varepsilon^N}\,dx - m\delta_{y_\varepsilon(t)}\Big\|_{C^{2*}} \le C\Omega_\varepsilon(t) + \mathcal{O}(\varepsilon^2),$$

which yields the assertion. □

Finally, we get a strengthened version of Lemma 6.1.

Lemma 6.4. *Let $\varepsilon_0 > 0$ and $T^*_\varepsilon > 0$ be as in Theorem 4.2. Then there exists a positive constant C such that*

$$\Big\|\frac{|\phi_\varepsilon(x,t)|^2}{\varepsilon^N}dx - m\delta_{x(t)}\Big\|_{C^{2*}} + \big\|p^A_\varepsilon(x,t)dx - m\xi(t)\delta_{x(t)}\big\|_{C^{2*}} \le C\Omega_\varepsilon(t) + \mathcal{O}(\varepsilon^2),$$

for every $t \in [0, T^*_\varepsilon)$ and $\varepsilon \in (0, \varepsilon_0)$. In particular, if $\|A\|_{C^2}$ is sufficiently small, we have

$$\Big\|\frac{|\phi_\varepsilon(x,t)|^2}{\varepsilon^N}dx - m\delta_{x(t)}\Big\|_{C^{2*}} + \big\|p^A_\varepsilon(x,t)dx - m\xi(t)\delta_{x(t)}\big\|_{C^{2*}} \le C\hat{\Omega}_\varepsilon(t) + \mathcal{O}(\varepsilon^2), \tag{6.6}$$

for every $t \in [0, T^*_\varepsilon)$ and $\varepsilon \in (0, \varepsilon_0)$.

Proof. Notice that, taking into account Lemma 6.1, Lemma 3.8 and Lemma 6.3, we get

$$\begin{aligned}\Big\|\frac{|\phi_\varepsilon(x,t)|^2}{\varepsilon^N}dx - m\delta_{x(t)}\Big\|_{C^{2*}} &\le \Big\|\frac{|\phi_\varepsilon(x,t)|^2}{\varepsilon^N}dx - m\delta_{y_\varepsilon(t)}\Big\|_{C^{2*}} \\ &\quad + m\big\|\delta_{y_\varepsilon(t)} - \delta_{x(t)}\big\|_{C^{2*}} \le C\Omega_\varepsilon(t) + \mathcal{O}(\varepsilon^2),\end{aligned}$$

for every $t \in [0, T^*_\varepsilon)$ and $\varepsilon \in (0, \varepsilon_0)$. In turn, we also get

$$\begin{aligned}&\big\|p^A_\varepsilon(x,t)dx - m\xi(t)\delta_{x(t)}\big\|_{C^{2*}} \le \big\|p^A_\varepsilon(x,t)dx - p^{A(x(t))}_\varepsilon(x,t)dx\big\|_{C^{2*}} \\ &+ \big\|p^{A(x(t))}_\varepsilon(x,t)dx - m\xi(t)\delta_{y_\varepsilon(t)}\big\|_{C^{2*}} + \big\|m\xi(t)\delta_{y_\varepsilon(t)} - m\xi(t)\delta_{x(t)}\big\|_{C^{2*}} \\ &\le \sup_{\|\varphi\|_{C^2}\le 1}\Big|\int [A(x) - A(x(t))]\varphi(x)\frac{|\phi_\varepsilon(x,t)|^2}{\varepsilon^N}dx\Big| + C\Omega_\varepsilon(t) + \mathcal{O}(\varepsilon^2) \\ &= \sup_{\|\varphi\|_{C^2}\le 1}\Big|\int [A(x) - A(x(t))]\varphi(x)\Big[\frac{|\phi_\varepsilon(x,t)|^2}{\varepsilon^N}dx - m\delta_{x(t)}\Big]dx\Big| + C\Omega_\varepsilon(t) + \mathcal{O}(\varepsilon^2) \\ &\le \sup_{\|\varphi\|_{C^2}\le 1}\|(A(x) - A(x(t)))\varphi(x)\|_{C^2}\Big\|\frac{|\phi_\varepsilon(x,t)|^2}{\varepsilon^N}dx - m\delta_{x(t)}\Big\|_{C^{2*}} + C\Omega_\varepsilon(t) + \mathcal{O}(\varepsilon^2) \\ &\le C\Big\|\frac{|\phi_\varepsilon(x,t)|^2}{\varepsilon^N}dx - m\delta_{x(t)}\Big\|_{C^{2*}} + C\Omega_\varepsilon(t) + \mathcal{O}(\varepsilon^2) \le C\Omega_\varepsilon(t) + \mathcal{O}(\varepsilon^2),\end{aligned}$$

for every $t \in [0, T^*_\varepsilon)$ and $\varepsilon \in (0, \varepsilon_0)$. This concludes the proof of the first assertion. Taking into account the definitions of $\Omega_\varepsilon(t)$ and $\rho^A_\varepsilon(t)$, inequality (6.6) is just a simple consequence. □

7. Proof of the main result concluded

In this section we will conclude the proof of the main result.

7.1. **The error estimate.** We now show that the quantity $\Omega_\varepsilon(t)$, introduced in (3.7), can be made small at the order $\mathcal{O}(\varepsilon^2)$, uniformly on finite time intervals, as $\varepsilon \to 0$.

Lemma 7.1. *There exists a positive constant $C = C(T_0)$ such that $\hat{\Omega}_\varepsilon(t) \le C(T_0)\varepsilon^2$, for all $\varepsilon \in (0, \varepsilon_0)$ and $t \in [0, T^*_\varepsilon)$. If in addition we assume that $\|A\|_{C^2} < \delta$ for some $\delta > 0$ small, then there exists a positive constant $C = C(T_0)$ such that $\Omega_\varepsilon(t) \le C(T_0)\varepsilon^2$, for all $\varepsilon \in (0, \varepsilon_0)$ and $t \in [0, T^*_\varepsilon)$.*

Proof. Taking into account Lemma 6.4, via identity (5.2) of Lemma 5.1, we obtain

$$
\begin{aligned}
&\Big| \int \frac{d}{dt} \Pi^1_\varepsilon(x,t)dx \Big| = \Big| \int \frac{\partial p^A_\varepsilon}{\partial t}(x,t)dx - m\dot{\xi}(t) \Big| \\
&= \Big| \int p^A_\varepsilon(x,t) \times B(x)dx + \int \nabla V(x) \frac{|\phi_\varepsilon(x,t)|^2}{\varepsilon^N} dx \\
&- m\nabla V(x(t)) - m\xi(t) \times B(x(t)) \Big| \\
&= \Big| \int p^A_\varepsilon(x,t) \times B(x)dx + \int \nabla V(x) \frac{|\phi_\varepsilon(x,t)|^2}{\varepsilon^N} dx \\
&- \int m\nabla V(x)\delta_{x(t)}dx - m \int \xi(t) \times B(x)\delta_{x(t)}dx \Big| \\
&\le \Big| \int \Big(p^A_\varepsilon(x,t) - m\xi(t)\delta_{x(t)} \Big) \times B(x)dx \Big| \\
&+ \Big| \int \nabla V(x) \Big(\frac{|\phi_\varepsilon(x,t)|^2}{\varepsilon^N} - m\delta_{x(t)} \Big) dx \Big| \\
&\le \|A\|_{C^3} \Big\| p^A_\varepsilon(x,t)dx - m\xi(t)\delta_{x(t)} \Big\|_{C^{2*}} \\
&+ \|V\|_{C^3} \Big\| \frac{|\phi_\varepsilon(x,t)|^2}{\varepsilon^N} dx - m\delta_{x(t)} \Big\|_{C^{2*}} \\
&\le C\hat{\Omega}_\varepsilon(t) + \mathcal{O}(\varepsilon^2),
\end{aligned}
$$

for all $\varepsilon \in (0, \varepsilon_0)$ and $t \in [0, T^*_\varepsilon)$. Hence, recalling Lemma 3.9, it follows that

$$
\begin{aligned}
\Big| \int \Pi^1_\varepsilon(x,t)dx \Big| &\le \Big| \int \Pi^1_\varepsilon(x,0)dx \Big| + \int_0^t \Big| \int \frac{d}{dt} \Pi^1_\varepsilon(x,\tau)dx \Big| d\tau \\
&\le \mathcal{O}(\varepsilon^2) + C \int_0^t \hat{\Omega}_\varepsilon(\tau)d\tau. \qquad (7.1)
\end{aligned}
$$

Let now $\varphi \in C^3(\mathbb{R}^N)$ with $\|\varphi\|_{C^3(\mathbb{R}^N)} \leq 1$. Then identity (5.1) and Lemma 6.4 yield

$$\begin{aligned}
&\Big|\int \frac{d}{dt}\Pi^2_\varepsilon(x,t)\varphi(x)dx\Big| = \Big|\int \varphi\frac{\partial}{\partial t}\frac{|\phi_\varepsilon(x,t)|^2}{\varepsilon^N}dx - m\nabla\varphi(x(t))\cdot\xi(t)\Big| \\
&= \Big| - \int \varphi(x)\,\mathrm{div}_x\, p^A_\varepsilon(x,t)dx - m\nabla\varphi(x(t))\cdot\xi(t)\Big| \\
&= \Big|\int \nabla\varphi(x)\cdot p^A_\varepsilon(x,t)dx - \int m\nabla\varphi(x)\cdot\xi(t)\delta_{x(t)}dx\Big| \\
&= \Big|\int \nabla\varphi(x)\cdot\Big(p^A_\varepsilon(x,t) - m\xi(t)\delta_{x(t)}\Big)dx\Big| \\
&\leq \|\varphi\|_{C^3}\big\|p^A_\varepsilon(x,t)dx - m\xi(t)\delta_{x(t)}\big\|_{C^{2*}} \leq C\hat{\Omega}_\varepsilon(t) + \mathcal{O}(\varepsilon^2),
\end{aligned}$$

for all $\varepsilon \in (0,\varepsilon_0)$ and $t \in [0,T^*_\varepsilon)$. Hence, by Lemma 3.9, it follows that

$$\begin{aligned}
\sup_{\|\varphi\|_{C^3}\leq 1}\Big|\int \Pi^2_\varepsilon(x,t)\varphi(x)dx\Big| &\leq \sup_{\|\varphi\|_{C^3}\leq 1}\Big|\int \Pi^2_\varepsilon(x,0)\varphi(x)dx\Big| \\
&\quad + \sup_{\|\varphi\|_{C^3}\leq 1}\int_0^t\Big|\int \frac{d}{dt}\Pi^2_\varepsilon(x,\tau)\varphi(x)dx\Big|d\tau \\
&\leq \mathcal{O}(\varepsilon^2) + C\int_0^t \hat{\Omega}_\varepsilon(\tau)d\tau,
\end{aligned} \tag{7.2}$$

for all $\varepsilon \in (0,\varepsilon_0)$ and $t \in [0,T^*_\varepsilon)$. Finally, again via identity (5.1) and Lemma 6.4,

$$\begin{aligned}
\big|\dot{\gamma}_\varepsilon(t)\big| &= \Big|m\xi(t) + \int x\chi(x)\mathrm{div}_x\, p^A_\varepsilon(x,t)dx\Big| \\
&= \Big|m\xi(t) - \int \nabla(x\chi(x))\cdot p^A_\varepsilon(x,t)dx\Big| \\
&= \Big|\int \nabla(x\chi(x))m\xi(t)\delta_{x(t)} - \int \nabla(x\chi(x))\cdot p^A_\varepsilon(x,t)dx\Big| \\
&\leq \|\nabla(x\chi(x))\|_{C^2}\big\|p^A_\varepsilon(x,t)dx - m\xi(t)\delta_{x(t)}\big\|_{C^{2*}} \leq C\hat{\Omega}_\varepsilon(t) + \mathcal{O}(\varepsilon^2),
\end{aligned}$$

for all $\varepsilon \in (0,\varepsilon_0)$ and $t \in [0,T^*_\varepsilon)$. This, recalling Lemma 3.9, yields

$$|\gamma_\varepsilon(t)| \leq \mathcal{O}(\varepsilon^2) + C\int_0^t \hat{\Omega}_\varepsilon(\tau)d\tau, \tag{7.3}$$

for all $\varepsilon \in (0,\varepsilon_0)$ and $t \in [0,T^*_\varepsilon)$. By collecting inequalities (7.1), (7.2) and (7.3), we get

$$\hat{\Omega}_\varepsilon(t) \leq \mathcal{O}(\varepsilon^2) + C\int_0^t \hat{\Omega}_\varepsilon(\tau)d\tau$$

for all $\varepsilon \in (0,\varepsilon_0)$ and $t \in [0,T^*_\varepsilon)$. Then, by Gronwall Lemma, we have $\hat{\Omega}_\varepsilon(t) \leq C(T_0)\varepsilon^2$, for all $\varepsilon \in (0,\varepsilon_0)$ and $t \in [0,T^*_\varepsilon)$. Finally, recalling the definitions of $\Omega_\varepsilon(t)$ and $\rho^A_\varepsilon(t)$ and exploiting again Lemma 6.4 concludes the proof. □

We are now ready to conclude the proof of Theorem 2.4.

Let $\delta > 0$ be as in Lemma 7.1. Let us prove the first part of Theorem 2.4.
We recall that the value of $T_0 > 0$ was fixed in the proof of Lemma 6.3 and it just depends on the data of the problem, such as V, A, m, N. Moreover, by virtue of Lemma 7.1 and by

the definition of T^*_ε (see the proof of Theorem 4.2), it follows that $T^*_\varepsilon = T_0$ for all $\varepsilon \in (0, \varepsilon_0)$, up to further reducing the value of ε_0. Hence $\Omega_\varepsilon(t) \le C(T_0)\varepsilon^2$ for all $\varepsilon \in (0, \varepsilon_0)$ and $t \in [0, T_0]$. Now, by Theorem 4.2 there exist two families of functions $\theta_\varepsilon : \mathbb{R}^+ \to [0, 2\pi)$ and $y_\varepsilon : \mathbb{R}^+ \to \mathbb{R}^N$ such that

$$\Big\| \phi_\varepsilon(\cdot, t) - e^{\frac{\mathrm{i}}{\varepsilon}(\xi(t)\cdot x + \theta_\varepsilon(t) + A(x(t))\cdot(x - x(t))} r\Big(\frac{x - y_\varepsilon(t)}{\varepsilon}\Big)\Big\|^2_{\mathbb{H}_\varepsilon} = \mathcal{O}(\varepsilon^2),$$

for all $t \in [0, T_0]$. On the other hand, by combining Lemma 6.3 with Lemma 7.1, it follows that $|x(t) - y_\varepsilon(t)| \le C\varepsilon^2$, for all $t \in [0, T_0]$ and $\varepsilon \in (0, \varepsilon_0)$. Then, taking into account the exponential decay of ∇r, we obtain

$$\Big\| r\Big(\frac{x - y_\varepsilon(t)}{\varepsilon}\Big) - r\Big(\frac{x - x(t)}{\varepsilon}\Big)\Big\|^2_{\mathbb{H}_\varepsilon} \le C\frac{|x(t) - y_\varepsilon(t)|^2}{\varepsilon^2} = \mathcal{O}(\varepsilon^2),$$

for all $t \in [0, T_0]$ and $\varepsilon \in (0, \varepsilon_0)$. Therefore, Theorem 2.4 holds true on the time interval $[0, T_0]$. Let us take $x(T_0)$ and $\xi(T_0)$ as new initial data in system (2.6) and the function

$$\phi_0^{\mathrm{new}}(x) := r\Big(\frac{x - x(T_0)}{\varepsilon}\Big) e^{\frac{\mathrm{i}}{\varepsilon}[A(x(T_0))\cdot(x - x(T_0)) + x\cdot\xi(T_0)]},$$

as a new initial data for problem (P). Whence, by the previous step of the proof, the approximation result holds on the interval $[T_0, 2T_0]$, and hence on an arbitrary finite time interval $[0, T]$, for $T > 0$.

In order to prove the second part of the statement of Theorem 2.4 one can follow the argument of [48] (essentially relying on [9]). Based upon the identity

$$\Big|\frac{\nabla v}{\mathrm{i}} - Av\Big|^2 = \frac{|p^A(v)|^2}{|v|^2} + |\nabla|v||^2, \qquad p^A(v) := \Im\big(\bar v(\nabla v - \mathrm{i}Av)\big),$$

the energy functional E_ε rewrites as

$$E_\varepsilon(t) = E_\varepsilon^{\mathrm{pot}}(t) + E_\varepsilon^{\mathrm{b}}(t) + E_\varepsilon^{\mathrm{k}}(t),$$

where we have set

$$E_\varepsilon^{\mathrm{pot}}(t) := \frac{1}{\varepsilon^N}\int V(x)|\phi_\varepsilon(x,t)|^2 dx, \qquad E_\varepsilon^{\mathrm{k}}(t) := \frac{\varepsilon^N}{2}\int \frac{|p_\varepsilon^A(x,t)|^2}{|\phi_\varepsilon(x,t)|^2} dx,$$
$$E_\varepsilon^{\mathrm{b}}(t) := \frac{1}{2\varepsilon^N}\int |\nabla|\phi_\varepsilon|(x,t)|^2 dx - \frac{1}{p+1}\frac{1}{\varepsilon^N}\int |\phi_\varepsilon(x,t)|^{2p+2} dx.$$

Then, following the steps of the proof of [48, Lemma 3.5] (on the basis of the quantitative estimate of the expansion of E_ε up to a error of $\mathcal{O}(\varepsilon^2)$, cf. Lemma 3.5), we get

$$0 \le E_\varepsilon^{\mathrm{b}}(|\phi_\varepsilon|) - E_\varepsilon^{\mathrm{b}}(r) \le C\hat\Omega_\varepsilon(t) + \mathcal{O}(\varepsilon^2), \tag{7.4}$$

$$0 \le E_\varepsilon^{\mathrm{k}}(t) - \frac{1}{2}\frac{\big|\int p_\varepsilon^A(x,t)\big|^2}{m} \le C\hat\Omega_\varepsilon(t) + \mathcal{O}(\varepsilon^2).$$

In turn, the second inequality easily yields

$$\int \Big| \varepsilon^{N/2}\frac{p_\varepsilon^A(x,t)}{|\phi_\varepsilon(x,t)|} - \frac{\big(\int p_\varepsilon^A(x,t)\big)}{m}\frac{|\phi_\varepsilon(x,t)|}{\varepsilon^{N/2}}\Big|^2 dx \le C\hat\Omega_\varepsilon(t) + \mathcal{O}(\varepsilon^2). \tag{7.5}$$

Once inequalities (7.4)-(7.5) holds true, the assertion can be proved by arguing as before. In fact, inequality (7.4) yields

$$\||\phi_\varepsilon| - r\Big(\frac{\cdot - y_\varepsilon(t)}{\varepsilon}\Big)\|^2_{\mathbb{H}_\varepsilon} \leq C\hat{\Omega}_\varepsilon(t) + \mathcal{O}(\varepsilon^2),$$

for some $y_\varepsilon(t) \in \mathbb{R}^N$. Instead, inequality (7.5) allows to prove inequality (6.6) of Lemma 6.4.

8. Conclusions

We have analyzed the soliton dynamics features of subcritical (with respect to global well-posedness) nonlinear Schrödinger equations in the semiclassical regime under the effects of an external electromagnetic field, showing that the solutions concentrate along a smooth curve $x(t) : \mathbb{R}^+ \to \mathbb{R}^N$ which is a parameterization of a solution of the classical Newton equation involving a conservative electric force $F_e = -\nabla V(x(t))$ as well as the contribution of the Lorenz force $F_b = -\dot{x}(t) \times B(x(t))$, being $B = \nabla \times A$ the magnetic field. The main results improves the results of [48], a recent contribution that the author discovered after completion of the paper. The technique is based upon the use of quantum (mass and energy for the PDE (P)) and classical ((2.7) for the ODE (1.4)) conservation laws, on the lines of an argument introduced in J. Bronski and R. Jerrard in 2000 in [9] making no use of a linearization procedure for the equation. On the other hand, the presence of the magnetic field introduces new difficulties that have to be handled. Finally, we wish to stress that our results are consistent with the current literature regarding the analysis of particular classes solutions, such as the standing waves.

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